Basics Equations for Filters Synthesis

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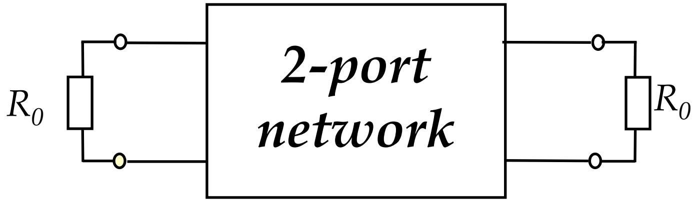
Scattering parameters for lumped-elements filters



For linear, lumped-element networks the scattering parameters can be always represented as a polynomial ratio of the complex frequency *p* $(p = \sigma + j\omega, \text{ with } \omega = 2\pi f)$:

$$S_{11} = \frac{N_{11}(p)}{D(p)}, \qquad S_{21} = S_{12} = \frac{N_{12}(p)}{D(p)}, \qquad S_{22} = \frac{N_{22}(p)}{D(p)}$$

Roots of N_{11} (N_{22}) : reflection zeros
Roots of N_{21} : transmission zeros
Roots of *D* : poles (natural frequencies)

The above polynomials are called *characteristic polynomials*

Properties of lossless networks

For synthesis purposes the 2-port network representing a filter is assumed lossless (i.e. composed by lossless elements)

The scattering matrix of a lossless network is **unitary:**

$$\mathbf{S}\cdot\tilde{\mathbf{S}}^{*}=\mathbf{U}_2 \quad \Rightarrow \quad \begin{aligned} S_{11}(p)\cdot S_{11}(p)^{*}+S_{21}(p)\cdot S_{21}(p)^{*}&=1\\ S_{22}(p)\cdot S_{22}(p)^{*}+S_{12}(p)\cdot S_{12}(p)^{*}&=1\\ S_{11}(p)\cdot S_{21}(p)^{*}+S_{12}(p)\cdot S_{22}(p)^{*}&=0 \end{aligned}$$

As a consequence, the following properties hold for the polynomials defining the scattering parameters (n is the filter order):

$$N_{11}(p)=N(p), \qquad N_{22}(p)=(-1)^{n}\,N(p)^{*}$$

$$N(p)\cdot N(p)^{*}+N_{21}(p)\cdot N_{21}(p)^{*}=D(p)\cdot D\;\;(p)^{*}$$

The latter expression is known as *Feldtkeller equation*

Properties of characteristic polynomials

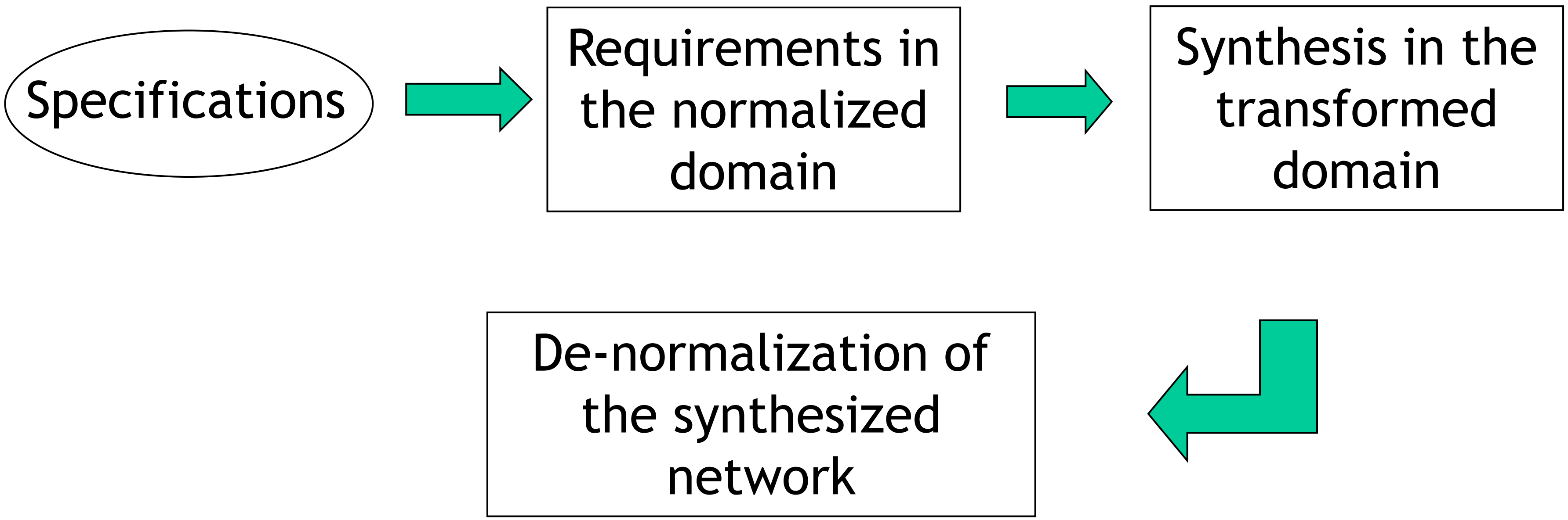
- The * define the paraconjugate operator: $Q(p)^* = Q^*(-p)$. The roots of $Q(p)^*$ are given by $-zQ^*$ (with zQ roots of Q)
- The roots of *D(p)* are real or complex conjugate pairs. The real part must be negative (strict *Hurwitz* polynomial)
- The roots of *N(p)* can be everywhere in the complex plane (if complex they occur as conjugate pairs). If they are on the imaginary axis $N_{22}(p)=N(p)$
- The roots of $N_{12}(p)$ can be on the imaginary axis (conjugate pair) or on the real axis (pairs with opposite values) or as a complex quad in the *p* plane
- The coefficients of all the polynomials are **real numbers.** Note that $Q(p)^* = Q(-p)$ in this case.

Steps of the synthesis process

- Assignment of selectivity specifications (Attenuation mask)
- Selection of a suitable *approximating function* and evaluation of its parameters in order to satisfy the specifications
- Evaluation of the characteristic polynomials from the approximating function
- Synthesis of the network from the polynomials

Synthesis in a transformed frequency domain

In order to simplify the design process, the synthesis can be performed in a **normalized low-pass domain**, analytically defined by *frequency transformations*. The synthesized network is then *de-normalized* to the band-pass frequency domain with suitable *circuit transformation*



Low-pass ←→ Band-pass transformation

Let be $s=\Sigma+j\Omega$ the normalized low-pass domain where the filter passband is defined Ω_B=-1↔+1.
The following equation relates the normalized domain with a pass-band domain suitably defined:

$$s = \frac{f_0}{B}\left(\frac{p}{f_0} + \frac{f_0}{p}\right)$$

f_0 and B are the frequency parameters defining the transformation:

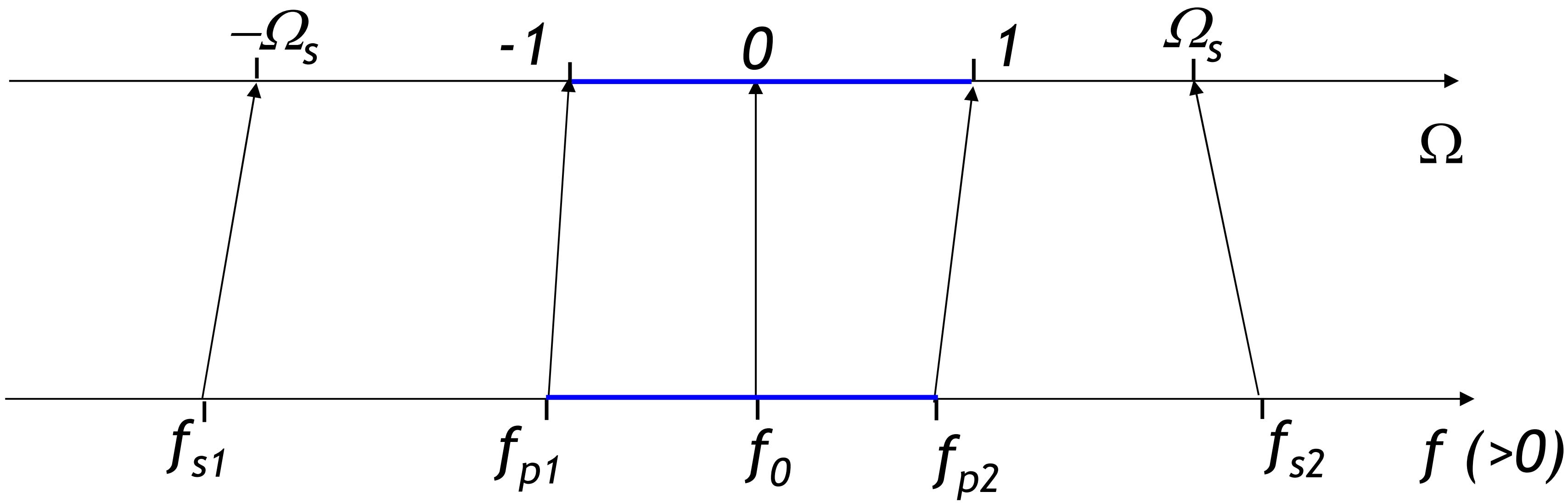
$$f_0 = \sqrt{f_{p1} \cdot f_{p2}}, \qquad B = f_{p2} - f_{p1}$$

The order of characteristic polynomials in the normalized domain is half of the order in the bandpass domain

Properties of the transformation (imaginary axis)

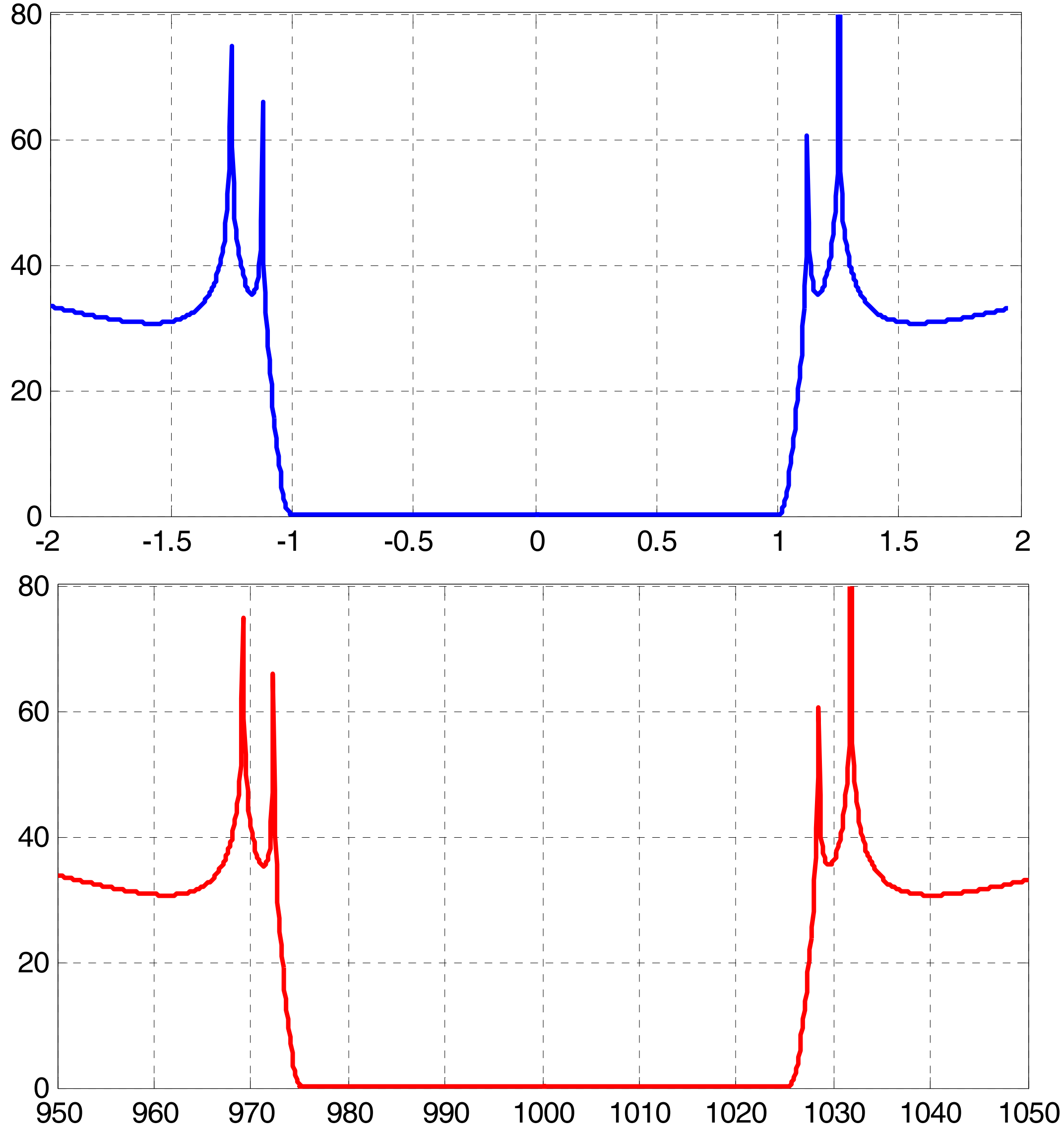
For $s=j\Omega$ and $p=j2\pi f$ it has

$$\Omega = \frac{f_0}{B}\left(\frac{f}{f_0} - \frac{f_0}{f}\right)$$



$$f_{p1} \cdot f_{p2} = f_{s1} \cdot f_{s2} = f_0^2$$

Conservation of response: **A(jΩ)=A(jω)**

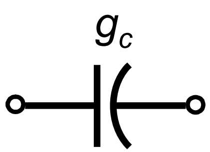


B=50 MHz

f_0=1000 MHz

Circuital property of the transformation: Conservation of Z and Y

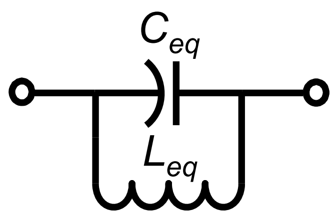
Low pass



$Y = s \cdot g_c$

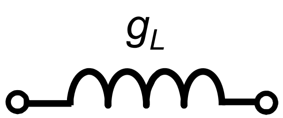
$$s = \frac{f_0}{B}\left(\frac{p}{f_0} + \frac{f_0}{p}\right)$$

Band pass

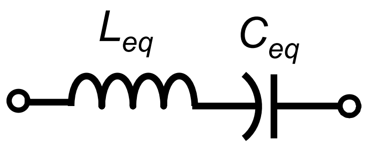


$$Y = p \cdot C_{eq} + \frac{1}{p \cdot L_{eq}} = \frac{f_0}{B}\left(\frac{p}{\omega_0} + \frac{\omega_0}{p}\right) \cdot g_C$$

$$C_{eq} = \frac{g_C}{2\pi B}, \qquad L_{eq} = \frac{1}{\omega_0^2 \cdot C_{eq}}$$



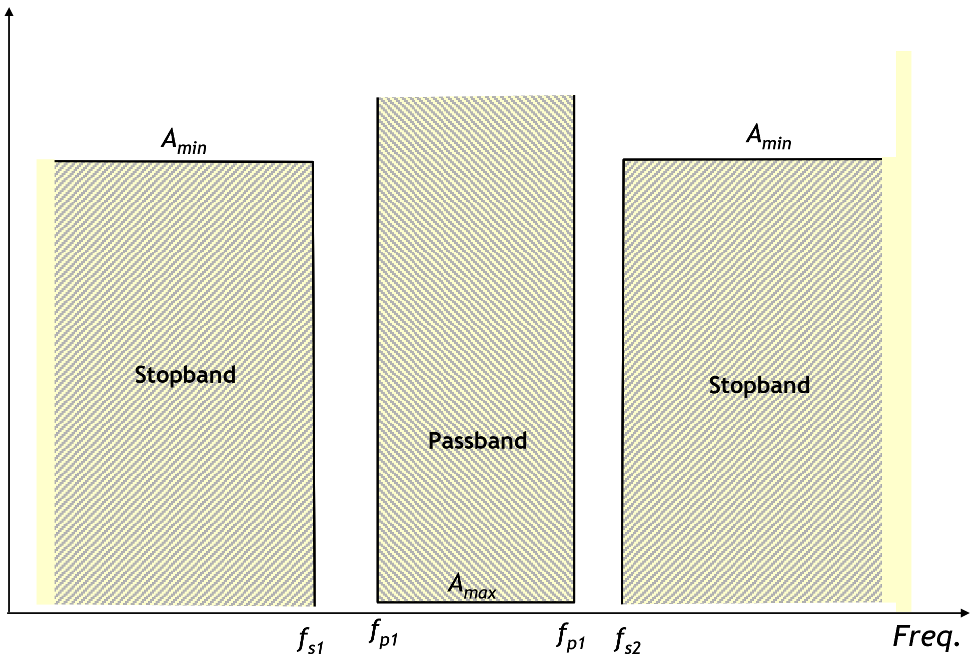
$Z = s \cdot g_L$



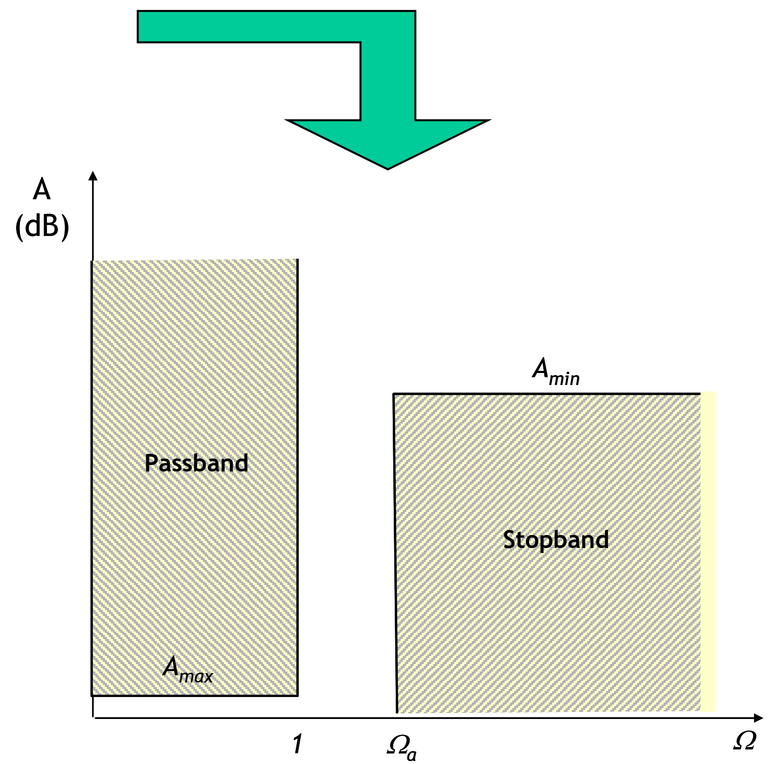
$$Z = p \cdot L_{eq} + \frac{1}{p \cdot C_{eq}} = \frac{f_0}{B}\left(\frac{p}{\omega_0} + \frac{\omega_0}{p}\right) \cdot g_L$$

$$L_{eq} = \frac{g_L}{2\pi B}, \qquad C_{eq} = \frac{1}{\omega_0^2 \cdot L_{eq}}$$

Specifications in the lowpass domain



$$f_{s,1} \cdot f_{s,2} = f_0^2$$



$$\Omega_a = \frac{f_0}{B}\left(\frac{f_{s,1}}{f_0} - \frac{f_0}{f_{s,1}}\right)$$

Approximation of the ideal Lowpass response

General expression for attenuation in the normalized domain:

$$A(\Omega) = 1 + \varepsilon'^2 C_n^2(\Omega) = \frac{1}{|S_{21}|^2} \qquad \varepsilon'^2 C_n^2(\Omega) = \frac{|S_{11}|^2}{|S_{21}|^2}$$

C_n is called *Characteristic Function* and defines the approximation of ideal filter response

Properties:

$-1 < C_n < 1$ for $-1 < \Omega < 1$

$|C_n| = 1$ for $\Omega = \pm 1$

$A(\pm 1) = 1 + \varepsilon'^2$

Parameters ε' and n

Maximum attenuation in passband: $A_{max} = 1+\varepsilon'^2$

$$A = \frac{1}{|S_{21}|^2} = \frac{1}{1-|S_{11}|^2}$$

Lossless networks: $|S_{11}|^2+|S_{21}|^2=1$
(Unitary S matrix)

$$1+\varepsilon'^2 = \frac{1}{1-|S_{11}|_{\max}^2} \quad \Rightarrow \quad \varepsilon'^2 = \frac{|S_{11}|_{\max}^2}{1-|S_{11}|_{\max}^2} = \frac{10^{-\frac{RL}{10}}}{1-10^{-\frac{RL}{10}}}$$

$$-20\cdot\log\left(|S_{11}|_{\max}\right) = \text{Return Loss } (RL)$$

The parameter n represents the *order* of the characteristic function and corresponds to the number of resonators in the de-normalized network

The characteristic function in s domain

Being $s=j\Omega$, it has:

$$C_n^2(\Omega) = C_n(j\Omega)\cdot C_n(-j\Omega) \quad \overset{\textit{analytic continuation}}{\Rightarrow} \quad C_n^2(s) = C_n(s)\cdot C_n(-s) = C_n(s)\cdot C_n^*(s)$$

$C_n(s)$ is an *analytic function* which can be in general expressed as the ratio of two monic polynomials:

$$C_n(s) = \frac{1}{\varepsilon'}\frac{S_{11}(s)}{S_{21}(s)} = \frac{\varepsilon}{\varepsilon'}\frac{F(s)}{P(s)}$$

The roots of *F(s)* are the *reflection zeros* (A=1). The roots of *P(s)* are the complex frequencies at which the attenuation becomes infinity (*transmission zeros*).
If *P(s)*=1 it has an *all-pole* characteristic function (Attenuation is infinite only for s=∞)

Characteristic polynomials in s domain

From the last expression of $C_n(s)$ the definition of the scattering parameters in *s* domain can be derived (all polynomials are monic):

$$S_{11}(s) = \frac{F(s)}{E(s)}, \qquad S_{21} = \frac{P(s)/\varepsilon}{E(s)}$$

Note that once F(s) and P(s) are known, *E(s)* is obtained by applying the unitary condition of S matrix:

$$P(s) \cdot P(-s) + \varepsilon^2 \left[F(s) \cdot F(-s) \right] = E(s) \cdot E(-s)$$

E(s) is found from the roots with negative real part of right-hand side (Hurwitz polynomial)

All-pole characteristic functions (P=cost)

- Butterworth function:

$$C_n(\Omega) = \Omega^n$$

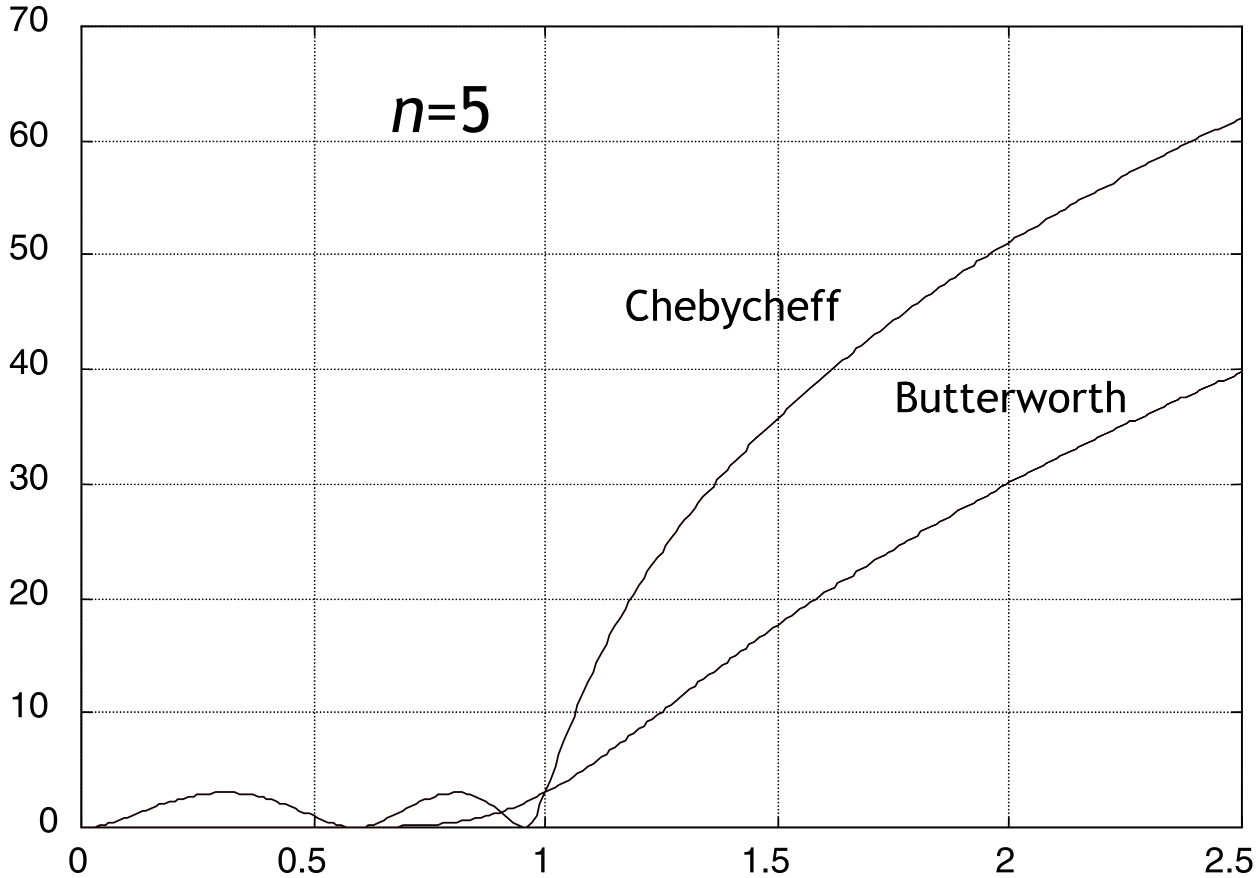
Main property: maximum flatness for Ω=0, monotonic for $|\Omega|>0$

- Chebycheff function:

$$C_n(\Omega) = \begin{cases} \cos\left(n \cdot \cos^{-1}(\Omega)\right) & |\Omega| \le 1 \\ \cosh\left(n \cdot \cosh^{-1}(\Omega)\right) & |\Omega| > 1 \end{cases}$$

Main property: oscillates between ±1 for $|\Omega|<1$ (*n* peaks), increases monotonically for $|\Omega|>1$

Comparison between Butterworth and Chebicheff



Attenuation of Chebycheff function is 6(n-1) dB larger than Butterworth function for $|\Omega|>>1$

Evaluation of ε and *n*

- <u>Assigned parameters</u>:
- Passband Return Loss in dB (RL)
- Filter bandwidth (B) and center frequency (f_0)
- Minimum attenuation (dB) in stopband (A_M)
- Frequencies at the beginning of stopband f_{s1}, f_{s2} with $f_{s1} \cdot f_{s2} = (f_0)^2$.

Evaluation of f_{s1} in the lowpass domain: $\Omega_s = \frac{f_0}{B}\left(\frac{f_{s1}}{f_0} - \frac{f_0}{f_{s1}}\right)$

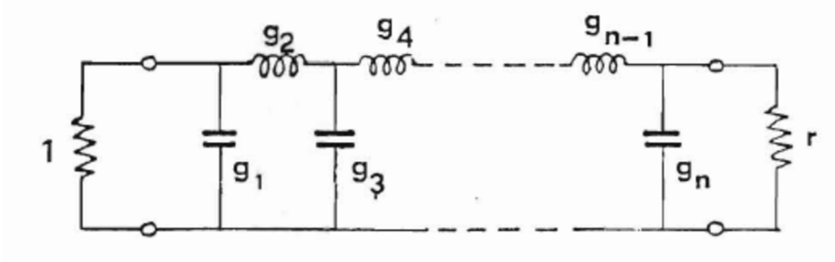
Evaluation of $\varepsilon' = \frac{\Gamma_m}{\sqrt{1-\Gamma_m^2}}$ with $\Gamma_m = 10^{-(RL/20)}$

$$n \geq \frac{A_M + RL}{20\log(\Omega_s)}$$ Butterworth characteristic

$$n \geq \frac{A_M + RL + 6}{20\log(\Omega_s) + 6}$$ Chebycheff characteristic

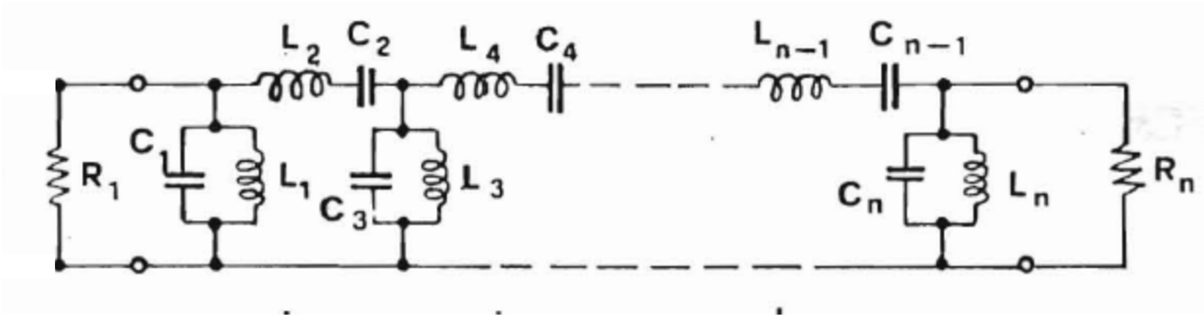
Normalized low-pass prototype (all-pole)

- Ladder network synthesized in the normalized domain assuming unitary reference load (generator side):



Lowpass

Bandpass



Bandpass de-normalization:

g_1, g_3, … → shunt resonators with $C_k = g_k / (R_1 \cdot 2\pi B)$

g_2, g_4, … → series resonators with $L_k = R_1 \cdot g_k / (2\pi B)$

Synthesis of lowpass prototype

- Butterworth characteristic

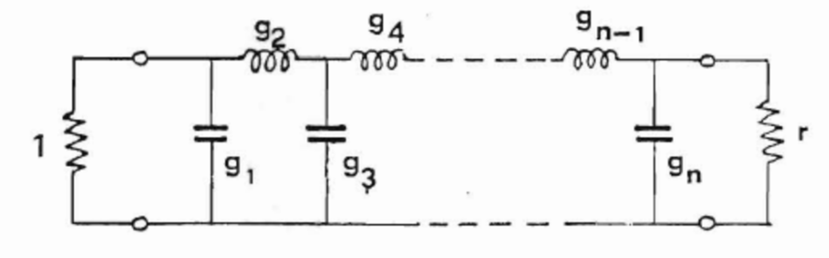
$$r_n = 1, \qquad g_q = 2a_q \sqrt[n]{\varepsilon'}, \qquad a_q = \sin\left(\frac{(2q-1)\pi}{2n}\right)$$

- Chebycheff characteristic

$$r_n = 1 \ \ (n \text{ odd}), \qquad r_n = \left[\sqrt{1+\varepsilon'^2} - \varepsilon'\right]^2 \ \ (n \text{ even})$$

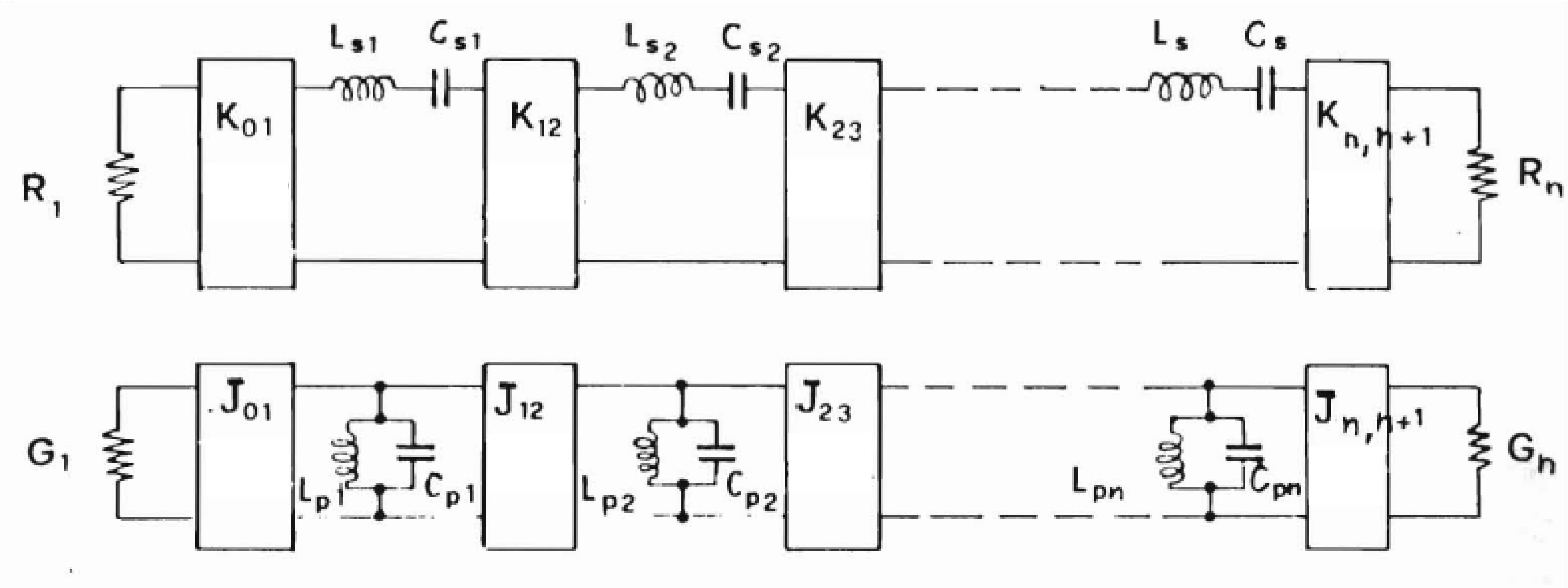
$$g_1 = \frac{2a_1}{\gamma}, \ \ g_q = \frac{4a_{q-1}\cdot a_q}{b_{q-1}\cdot g_{q-1}}, \qquad a_q = \sin\left(\frac{(2q-1)\pi}{2n}\right)$$

$$\gamma = \sinh\left(\frac{1}{2n}\ln\left(\frac{\sqrt{1+\varepsilon'^2}+1}{\sqrt{1+\varepsilon'^2}-1}\right)\right), \quad b_q = \gamma^2 + \sin^2\left(\frac{q\pi}{n}\right)$$



De-normalized (bandpass) network with only series or shunt resonators

For microwave frequencies implementation is requested to have only one type of resonator (series or shunt). This is obtained with the introduction of impedance (admittance) inverters



General design equation

Introduction of inverter determines additional degrees of freedom which allows the arbitrary assignments of some parameters (inverters or resonators parameter).
The condition to be satisfied are expressed by the following equations:

Series Network

$$K_{01} = \sqrt{\frac{B}{f_0} \frac{\omega_0 L_{s,1}}{g_1} R_1}$$

$$K_{q,q+1} = \frac{B}{f_0} \sqrt{\frac{\omega_0 L_{s,q} \cdot \omega_0 L_{s,q+1}}{g_q \cdot g_{q+1}}}$$

Shunt Network

$$J_{01} = \sqrt{\frac{B}{f_0} \frac{\omega_0 C_{s,1}}{g_1} G_1}$$

$$J_{q,q+1} = \frac{B}{f_0} \sqrt{\frac{\omega_0 C_{s,q} \cdot \omega_0 C_{s,q+1}}{g_q \cdot g_{q+1}}}$$

The bandpass network is assumed symmetric respect the central inverter (*n* even) or resonator (*n* odd)

Universal coupling parameters

Coupling coefficient:

$$k_{q,q+1} = \frac{K_{q,q+1}}{\sqrt{\omega_0 L_{s,q} \cdot \omega_0 L_{s,q+1}}} = \frac{J_{q,q+1}}{\sqrt{\omega_0 C_{s,q} \cdot \omega_0 C_{s,q+1}}} = \frac{B}{f_0}\sqrt{\frac{1}{g_q \cdot g_{q+1}}}$$

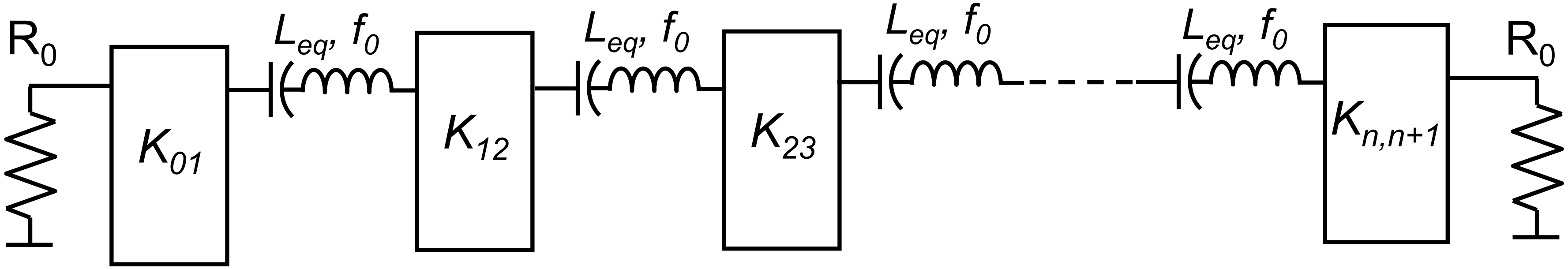
External Q:

$$Q_E = \frac{\omega_0 L_{s,1}}{K_{01}^2 / R_1} = \frac{\omega_0 C_{s,1}}{J_{01}^2 / G_1} = \frac{g_1}{B/f_0}$$

$\omega_0 L_{s,1}$ = Equivalent reactance of series resonators

$\omega_0 C_{s,1}$ = Equivalent susceptance of shunt resonators

In-line filters with all-equal resonators



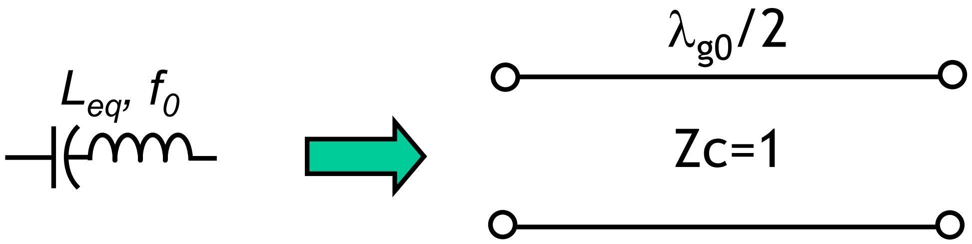
From the general design equations:

$$K_{q,q+1} = k_{q,q+1} \cdot \left(\omega_0 L_{eq}\right) \qquad K_{01} = K_{n,n+1} = \sqrt{\frac{\omega_0 L_{eq}}{Q_E} R_1}$$

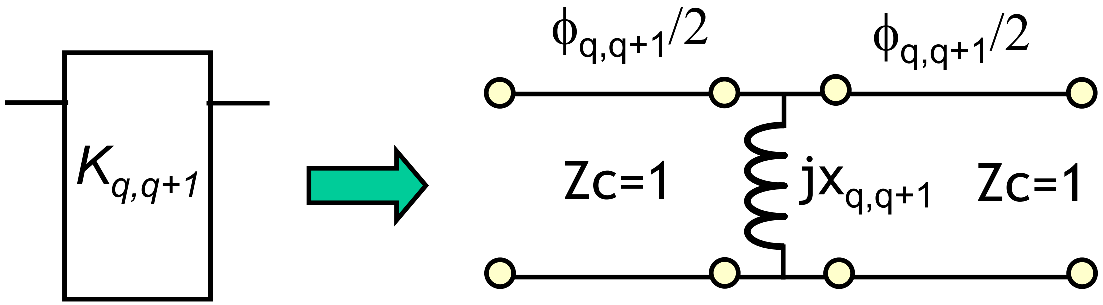
with:

$$k_{q,q+1} = \frac{B}{f_0}\sqrt{\frac{1}{g_q \cdot g_{q+1}}} \qquad Q_E = \frac{g_1}{B/f_0}$$

$\lambda_{g0}/2$ waveguide resonators coupled with inductive reactances

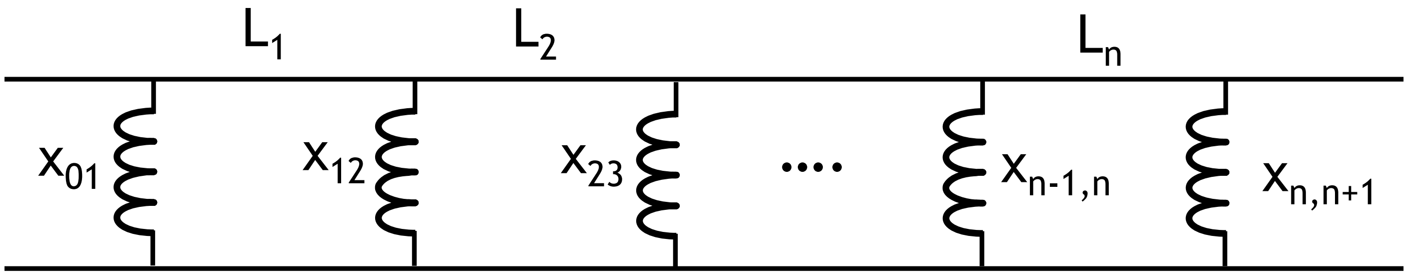


$$\omega_0 L_{eq} = \frac{\pi}{2} Z_C \left(\frac{\lambda_{g0}}{\lambda_0} \right)^2$$



$$x_{q,q+1} = \frac{\left(K_{q,q+1} / Z_c \right)}{1 - \left(\frac{K_{q,q+1}}{Z_C} \right)^2}$$

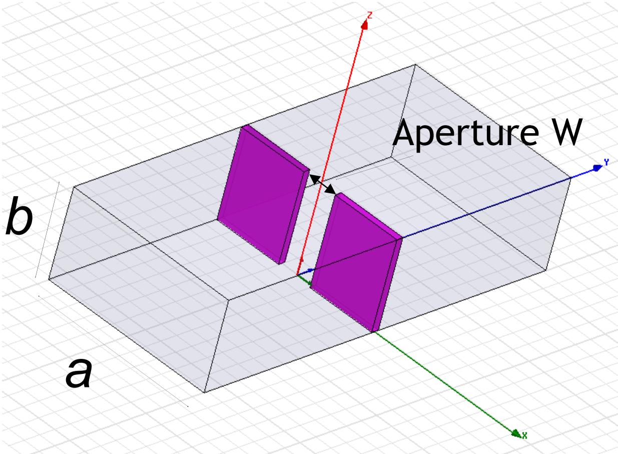
$$\phi_{q,q+1} = -\tan^{-1}\left(2x_{q,q+1} \right)$$

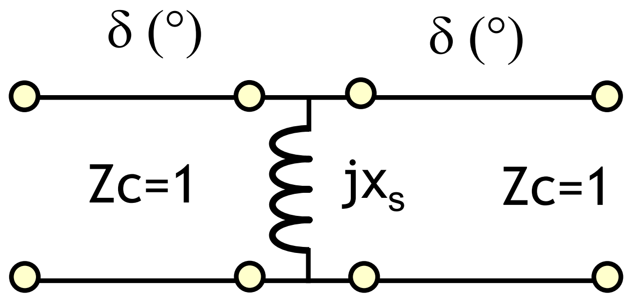


$$L_q = \frac{1}{\beta_{g0}} \left[\pi + \frac{\phi_{q-1,q} + \phi_{q,q+1}}{2} \right]$$

Inductive reactances in rectangular waveguide

- Iris with finite thickness



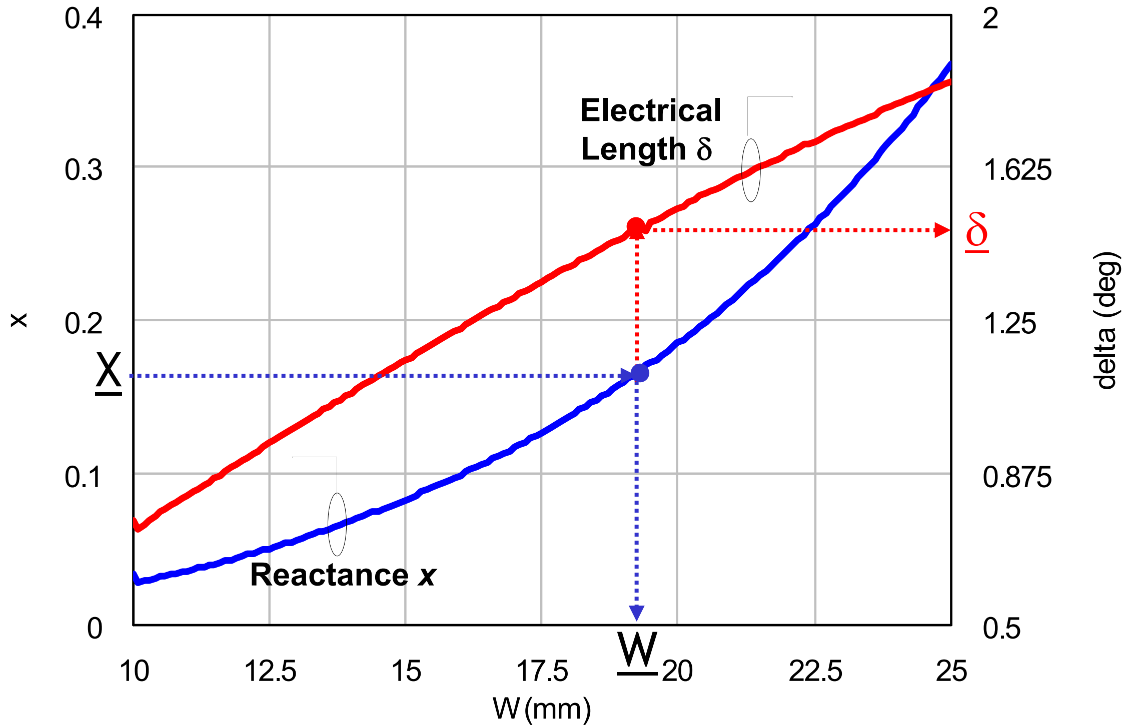


Equivalent circuit (reference sections: symmetry axis)

$$j\frac{X}{Y_c} = -\frac{1}{2}\frac{S_{12}}{S_{11}}, \qquad \exp(-j2\delta) = (S_{12} - S_{11})$$

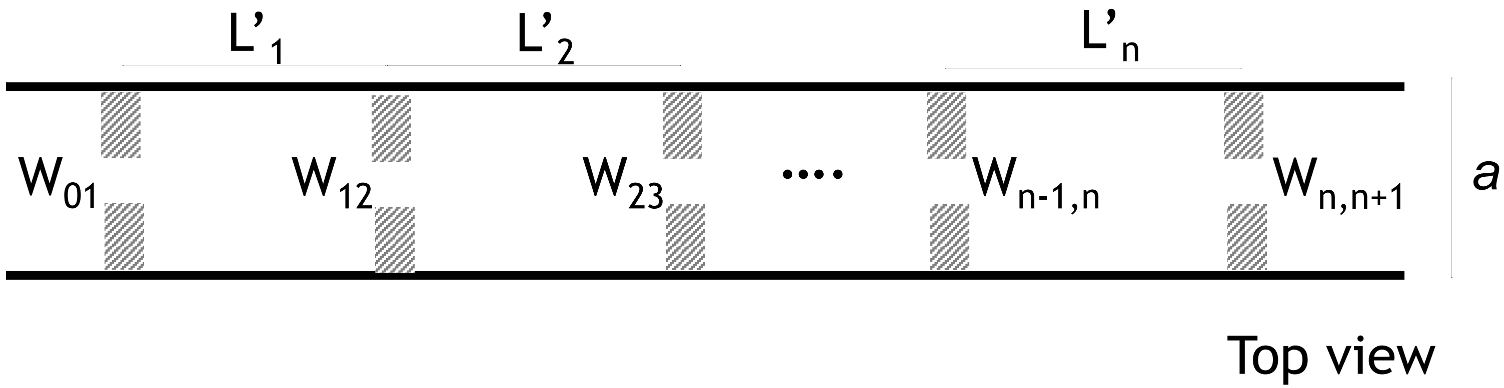
S_{11}, S_{12}: scattering parameters (computed numerically with EM simulators)

Evaluation of iris equivalent parameters



- With the previous formulas, the curves of *x* and δ vs. W are drawn (see the graph above).
- For the required value **X** the aperture **W** is first obtained through the blue curve. The corresponding electrical length δ is derived from the red curve

Filter dimensioning



The apertures $W_{i,i+1}$ are evaluated as explained in the previous slide from the $K_{i,i+1}$.

The separation of the iris are obtained from the lengths L_q corrected with the irises parameters $\delta_{q,q+1}$:

$$L'_q = L_q - \frac{1}{\beta_{g0}}\left(\delta_{q-1,q} + \delta_{q,q+1}\right)$$

Example of design

Specifications:

f_0=4 GHz, B=40 MHz

Return Loss in passband ≥26 dB

A_{s1} ≥ 45 dB for f>4.05 GHz

A_{s2} ≥ 60 dB for f>3.9 GHz

Chebycheff characteristic

Being f_{s1} and f_{s2} not geometrically symmetric, the requested order *n* of the filter is obtained by selecting the larger between the following ones:

$$n_1 \geq \frac{A_{s1} + RL + 6}{20\log(\Omega_{s1}) + 6} = 5.54 \qquad n_2 \geq \frac{A_{s2} + RL + 6}{20\log(\Omega_{s2}) + 6} = 4.58$$

***n*=6**

Step 1: Evaluation of equivalent circuit param.

Selected waveguide: a=50mm, b=25mm

Waveguide parameters:

Cutoff frequency: $f_c=v/2a$=3 GHz

$$\lambda_{g_0} = \frac{\lambda_0}{\sqrt{1-\left(f_c/f_0\right)^2}} = 113.3mm, \qquad \left(\frac{\lambda_{g_0}}{\lambda_0}\right)^2 = 2.286$$

Prototype normalized coefficients:

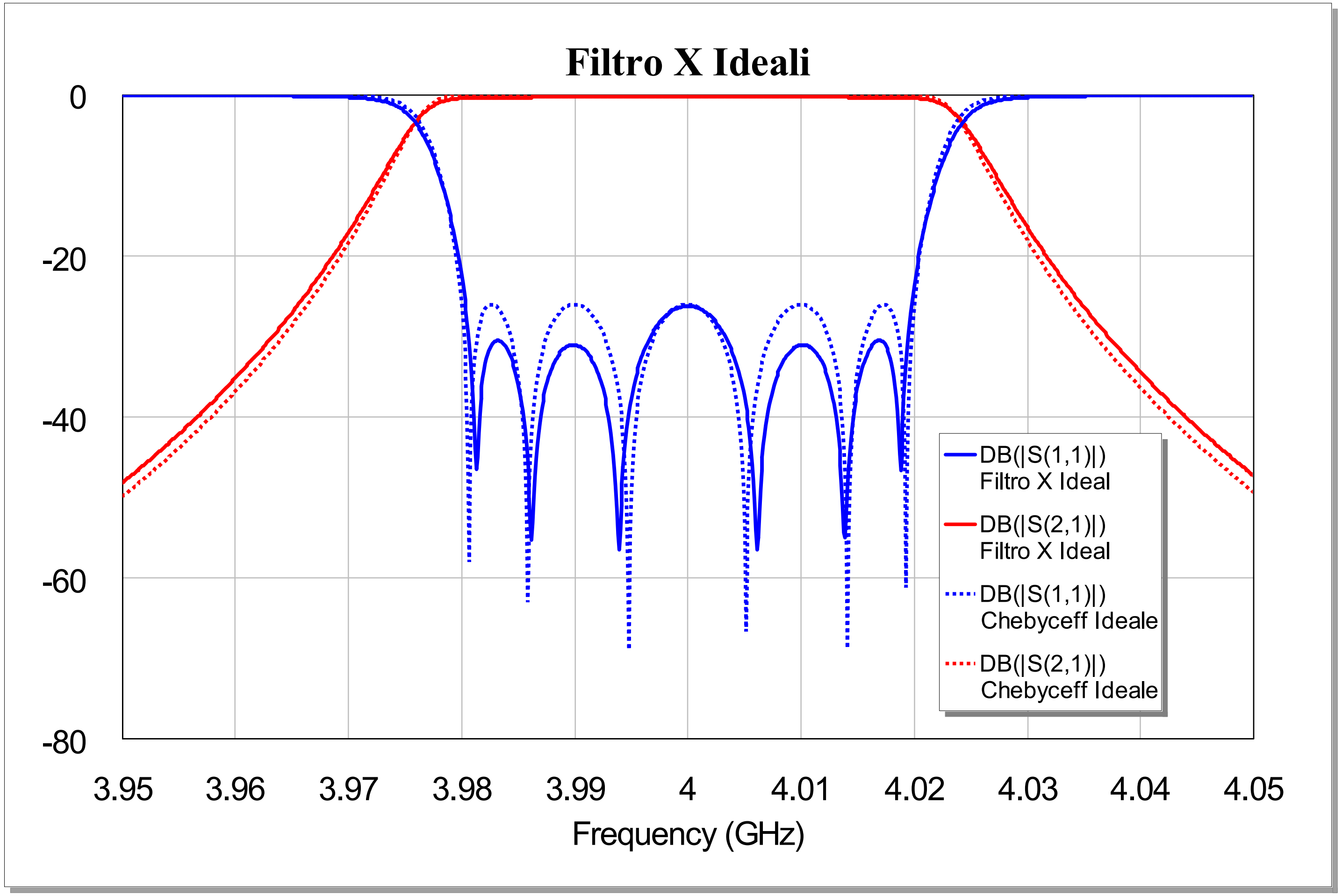
g_q={0.7919, 1.3649, 1.7002, 1.5379, 1.5089, 0.7163}

Coupling reactances and resonator lengths:

$x_{q,q+1}$={0.223, 0.0345, 0.0235, 0.022, 0.0235,0.0345,0.223}

L_q ={52.21, 55.56, 55.78, 55.78, 55.56, 52.21}

Filter response (ideal inductive reactance)

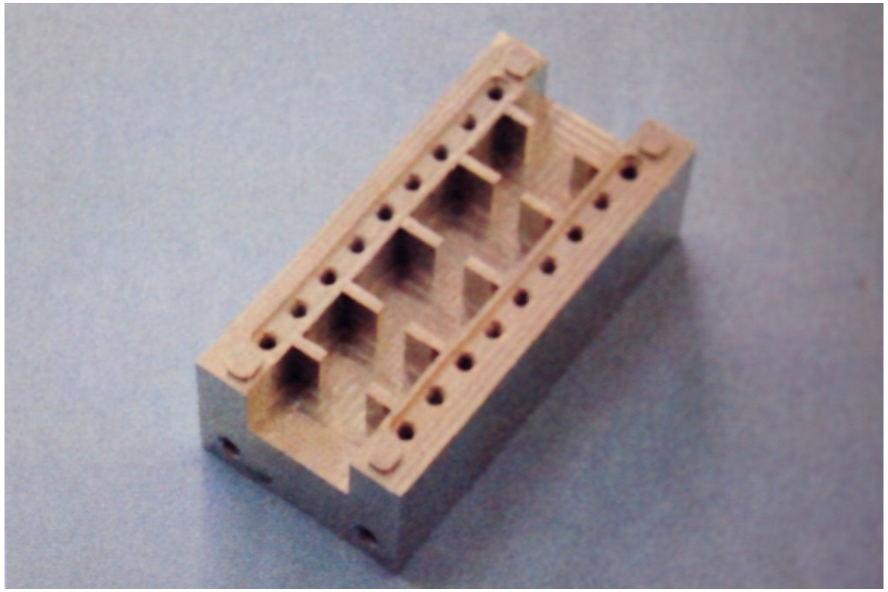


The reactances are assumed constant with *f*

Step 2: Irises dimensioning

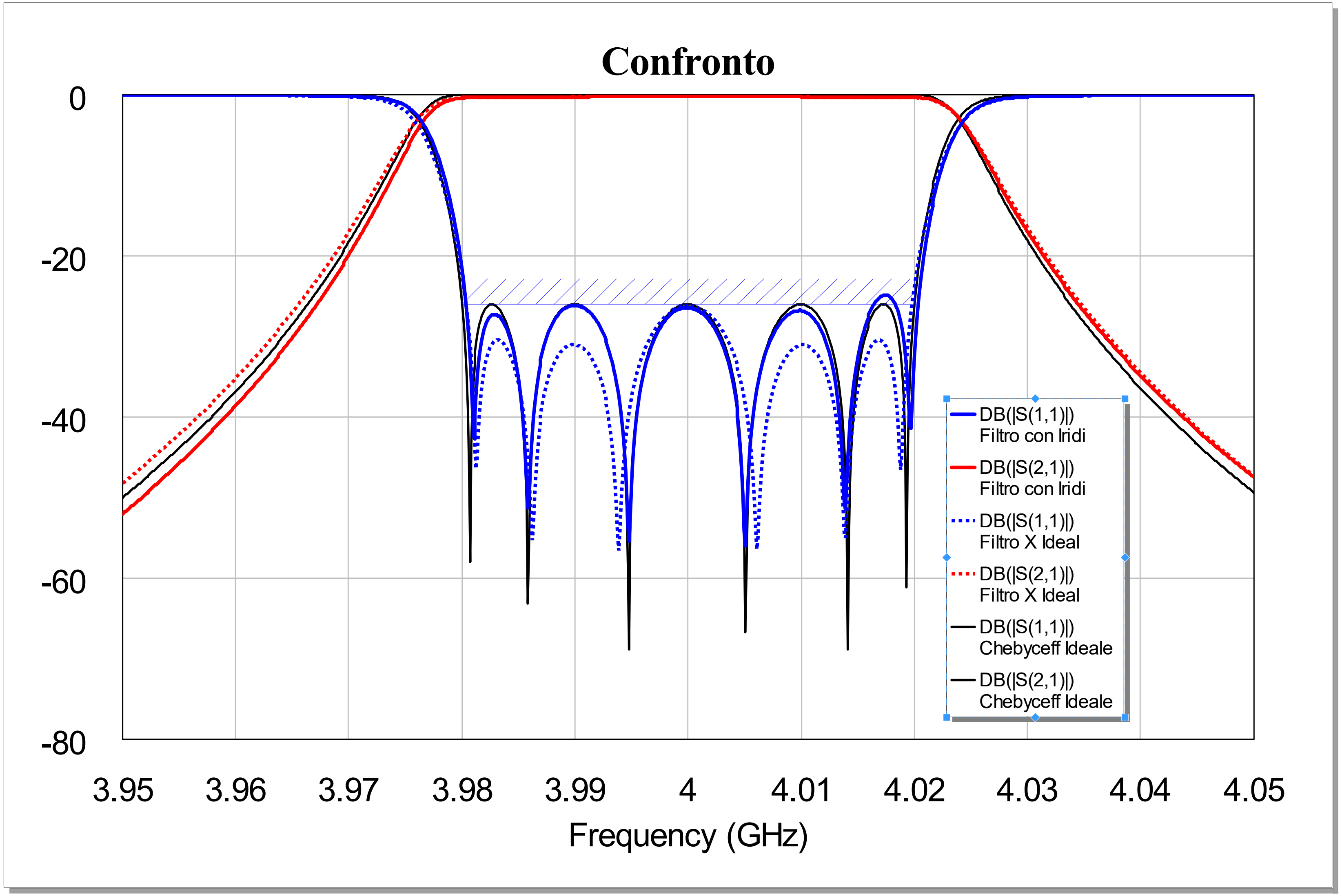
$W_{q,q+1}$={21.31, 10.85, 9.419, 9.213, 9.419, 10.85, 21.31} mm
$\delta_{q,q+1}$ ={0.507, 0.2536, 0.2145, 0.209, 0.2145, 0.2536,0.507 } mm

Corrected lengths of resonators:
L'_q={51.45, 55.09, 55.36, 55.36, 55.09, 51.45} mm

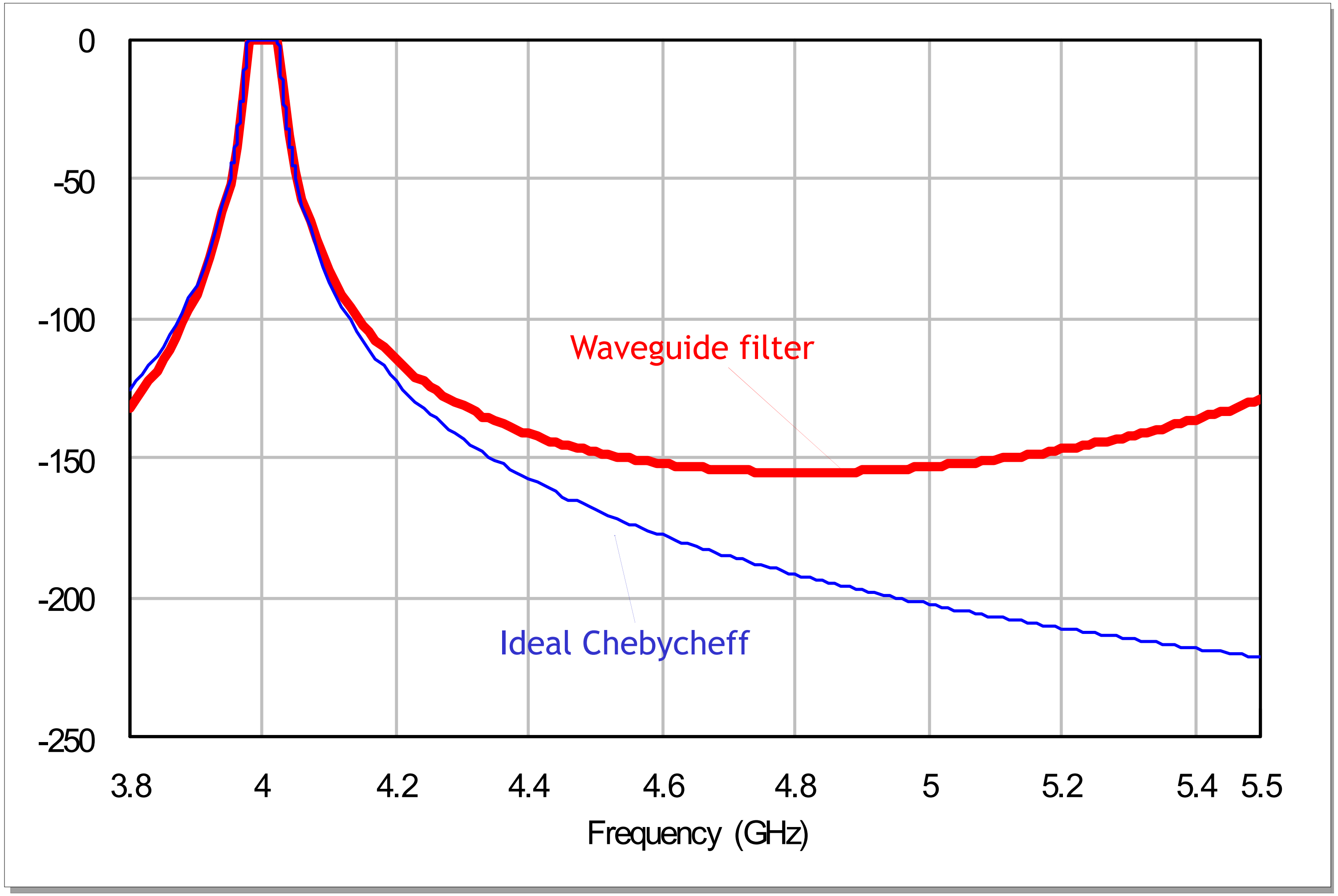


Example of fabricated device (4 resonators sample)

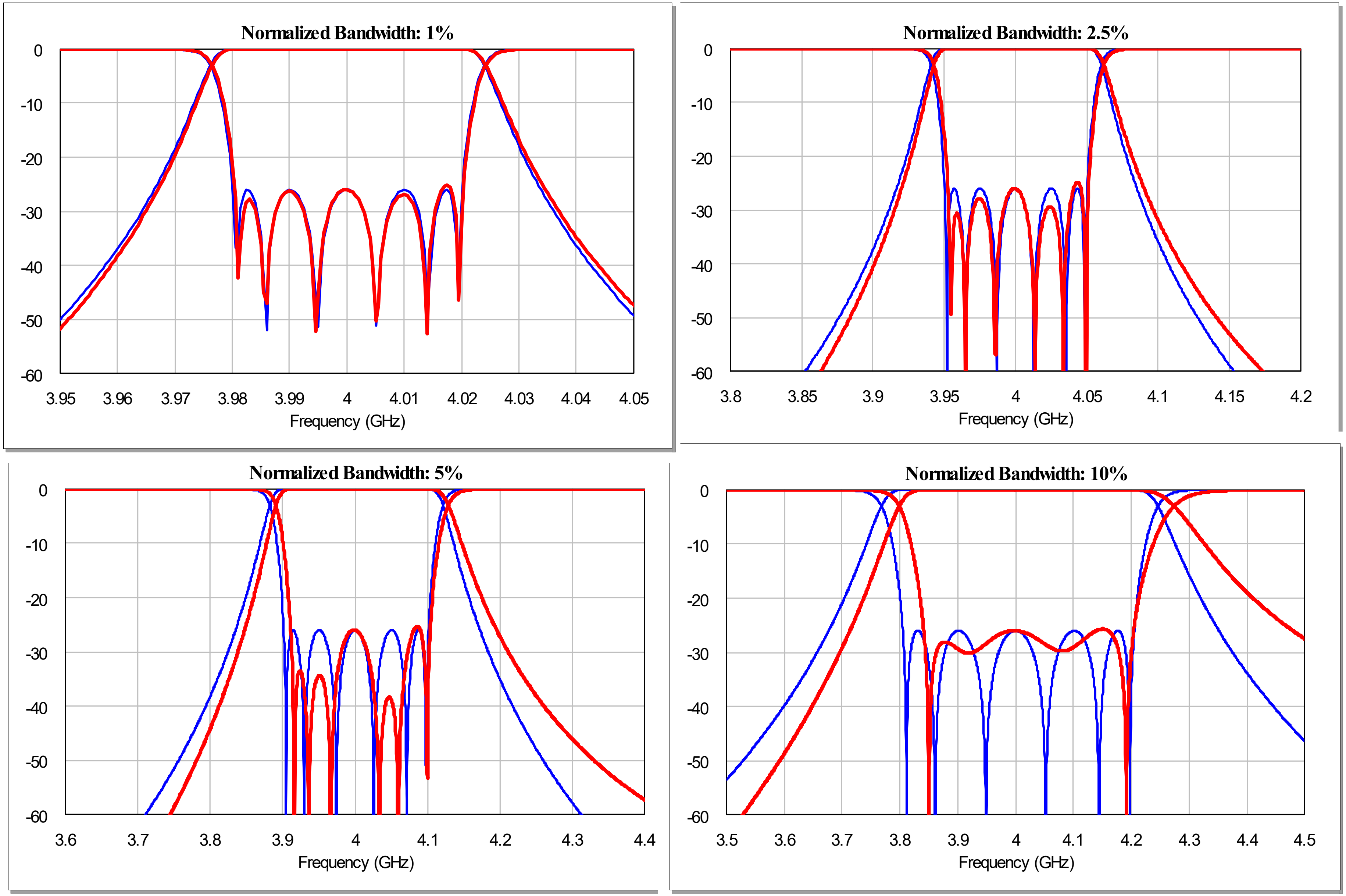
Computed filter response (Mode Matching)



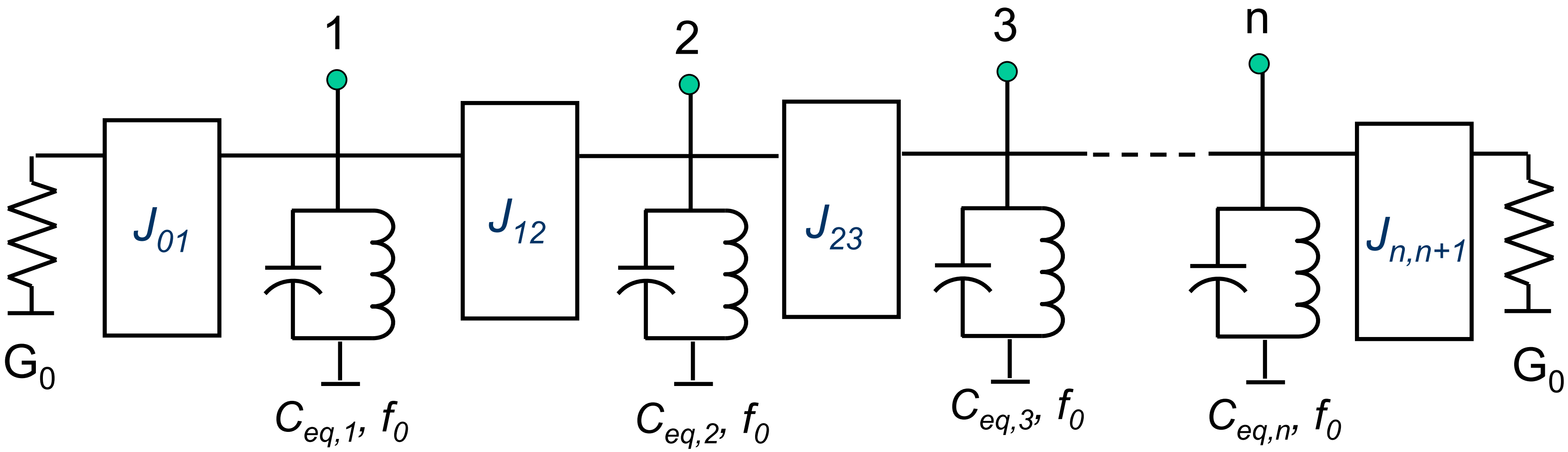
Broad frequency range response



Effect of filter bandwidth on accuracy



Multi-port simulation of the filter



Extraction of matrix Y (EM simulation):

$y_{i,i}=jB_{ris,I}$, $y_{i,i+1}=jJ_{i,i+1}$

$$B_{ris,i}(\omega)=\omega_0 C_{eq,i}\left(\frac{f}{f_0}-\frac{f_0}{f}\right) \quad \Rightarrow \quad \omega_0 C_{eq,i}=B_{eq,i}=\frac{1}{2}\omega_0 \frac{\partial \operatorname{Im}\left(y_{i,i}(\omega)\right)}{\partial \omega}\bigg|_{\omega=\omega_0}$$

$$J_{i,i+1}=\operatorname{Im}\left(y_{i,i+1}(\omega_0)\right)$$

Extraction of Universal Parameters Y matrix

$f_{0,i}$ is the frequency where $B_{i,i}$=0

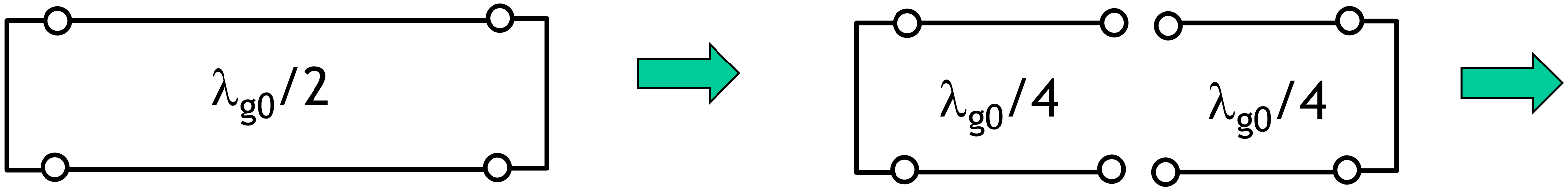
$$Q_{ext,1} = \frac{B_{\mathrm{eq},1}}{J_{01}^2} G_0 = \frac{\left[\frac{1}{2} \omega_0 \left. \frac{\partial \operatorname{Im}\left(y_{11}(\omega) \right)}{\partial \omega} \right|_{\omega=\omega_0} \right]}{\operatorname{Re}\left[y_{11}(\omega_0) \right]},$$

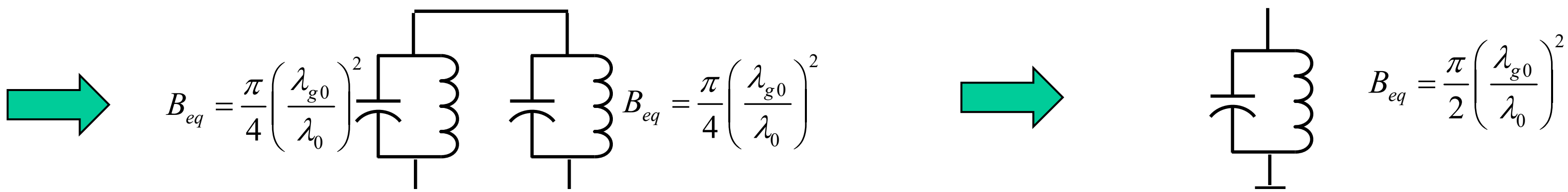
$$k_{i,i+1} = \frac{J_{i,i+1}}{\sqrt{B_{eq,i} \cdot B_{eq,j}}} = \frac{\operatorname{Im}\left(y_{i,i+1}(\omega_0) \right)}{\frac{1}{2} \omega_0 \sqrt{\frac{\partial \operatorname{Im}\left(y_{i,i}(\omega_0) \right)}{\partial \omega} \frac{\partial \operatorname{Im}\left(y_{j,j}(\omega_0) \right)}{\partial \omega}}}$$

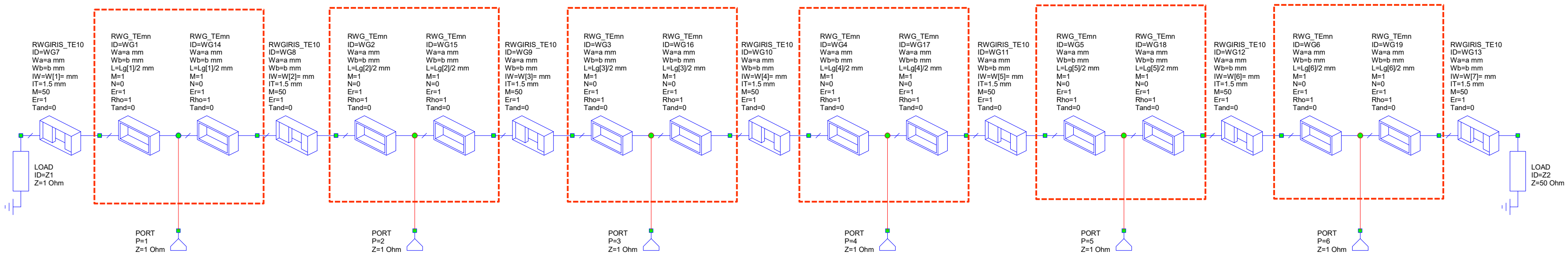
Example: waveguide filter seen before

Note

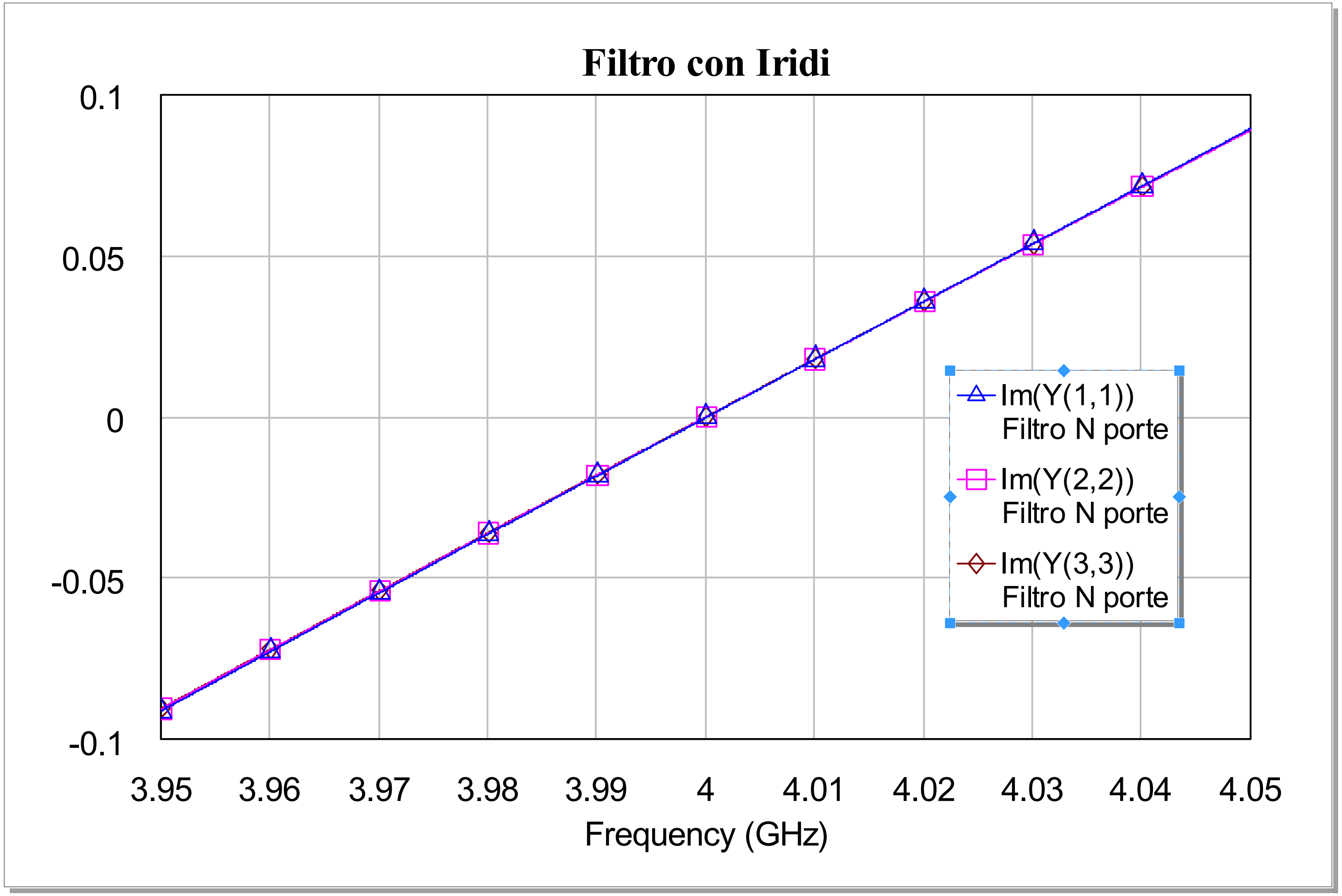
A short-circuited waveguide resonator of length $\lambda_{g0}/2$ observed at the center shows a shunt resonance:



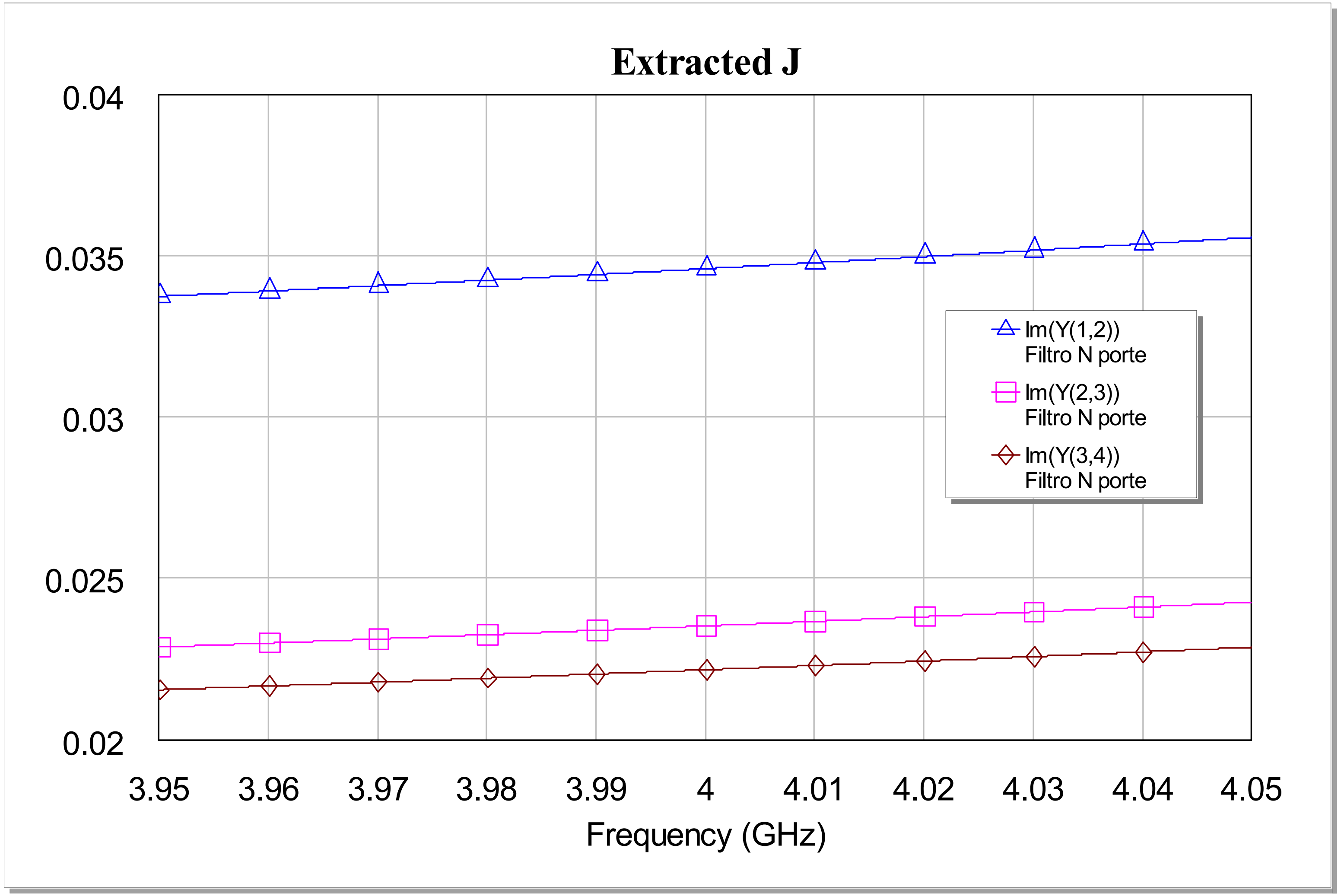




Extracted $B_{i,i}$



Extracted $J_{i,i+1}$



Computed Universal Parameters

Y11 = Filtro N porte:Y(1,1)
Y22 = Filtro N porte:Y(2,2)
Y33 = Filtro N porte:Y(3,3)

Y12 = Filtro N porte:Y(1,2)
Y23 = Filtro N porte:Y(2,3)
Y34 = Filtro N porte:Y(3,4)

F=_FREQ*1e-6

B11=imag(Y11)	J12=imag(Y12)
B22=imag(Y22)	J23=imag(Y23)
B33=imag(Y33)	J34=imag(Y34)

I1=find_index(B11,0)	F[I1]: 4000	I0=find_index(F,4000)
I2=find_index(B22,0)	F[I2]: 4000	
I3=find_index(B33,0)	F[I3]: 4000	

Beq1=0.5*der(B11,F)	
Beq1=Beq1[I1]*F[I1]	Beq1: 3.611
Beq2=0.5*der(B22,F)	
Beq2=Beq2[I2]*F[I2]	Beq2: 3.59
Beq3=0.5*der(B33,F)	
Beq3=Beq3[I3]*F[I3]	Beq3: 3.588

		Synthesized:
Qext=Beq1/real(Y11[I1])	1/Qext: 0.01271	k01= 0.01263
k12=J12[I0]/sqrt(Beq1*Beq2)	k12: 0.00961	k12=0.0096189
k23=J23[I0]/sqrt(Beq3*Beq2)	k23: 0.006554	k23=0.0065646
k34=J34[I0]/sqrt(Beq3*Beq3)	k34: 0.006176	k34=0.0061843