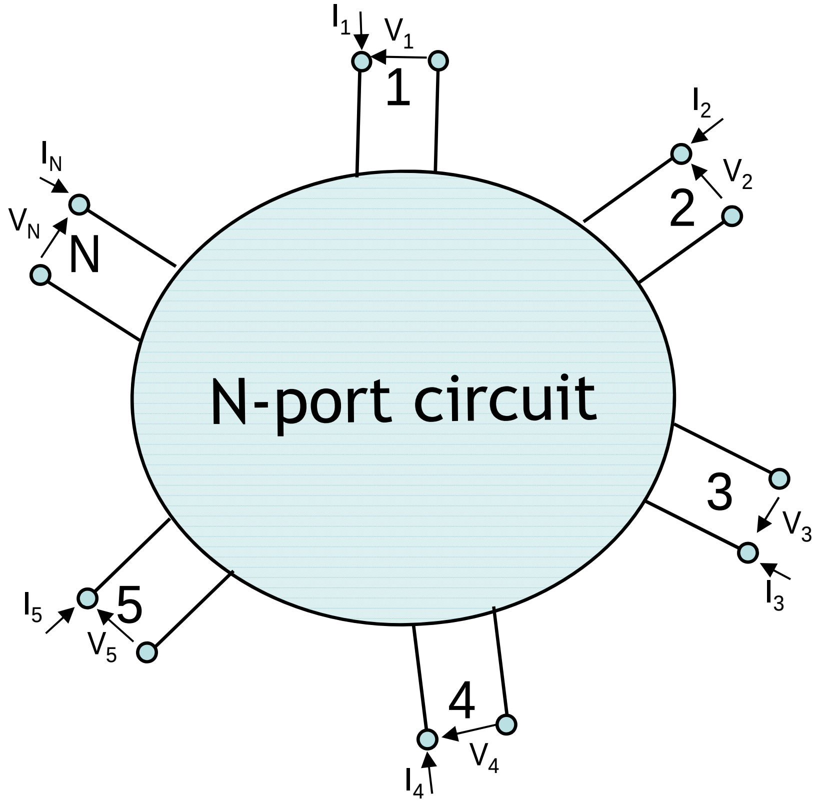
Matrix Characterization of Electrical Networks

Matrix modeling of a n-port linear circuit



$$\mathbf{V} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ . \\ v_N \end{bmatrix} \qquad \mathbf{I} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ . \\ i_N \end{bmatrix}$$

Assuming the network linear, the superposition principle allows the evaluation of the voltages at the ports, for a given vector of imposed currents (or vice versa). The matrices Z and Y are then introduced:

$$\mathbf{V} = \mathbf{Z} \cdot \mathbf{I} \qquad \mathbf{I} = \mathbf{Y} \cdot \mathbf{V}$$

Z = Impedance Matrix

Y = Admittance Matrix

Definition of **Z** elements

$$v_1 = z_{1,1}i_1 + z_{1,2}i_2 + z_{1,3}i_3 + \ldots + z_{1,N}i_N$$

$$v_2 = z_{2,1}i_1 + z_{2,2}i_2 + z_{2,3}i_3 + \ldots + z_{2,N}i_N$$

....

$$v_N = z_{N,1}i_1 + z_{N,2}i_2 + z_{N,3}i_3 + \ldots + z_{N,N}i_N$$

Independent (impressed) parameters**: Currents at the ports**

$z_{i,i}$= Impedance at port *i* with all the others open circuited

$z_{i,j}$= Ratio of the voltage at port *i* e the impressed current at port *j* with all the ports open circuited.

Definition of **Y** elements

$$i_1 = y_{1,1}v_1 + y_{1,2}v_2 + y_{1,3}v_3 + \ldots + y_{1,N}v_N$$

$$i_2 = y_{2,1}v_1 + y_{2,2}v_2 + y_{2,3}v_3 + \ldots + v_{2,N}v_N$$

$$\ldots\ldots\ldots$$

$$i_N = y_{N,1}v_1 + y_{N,2}v_2 + y_{N,3}v_3 + \ldots + y_{N,N}v_N$$

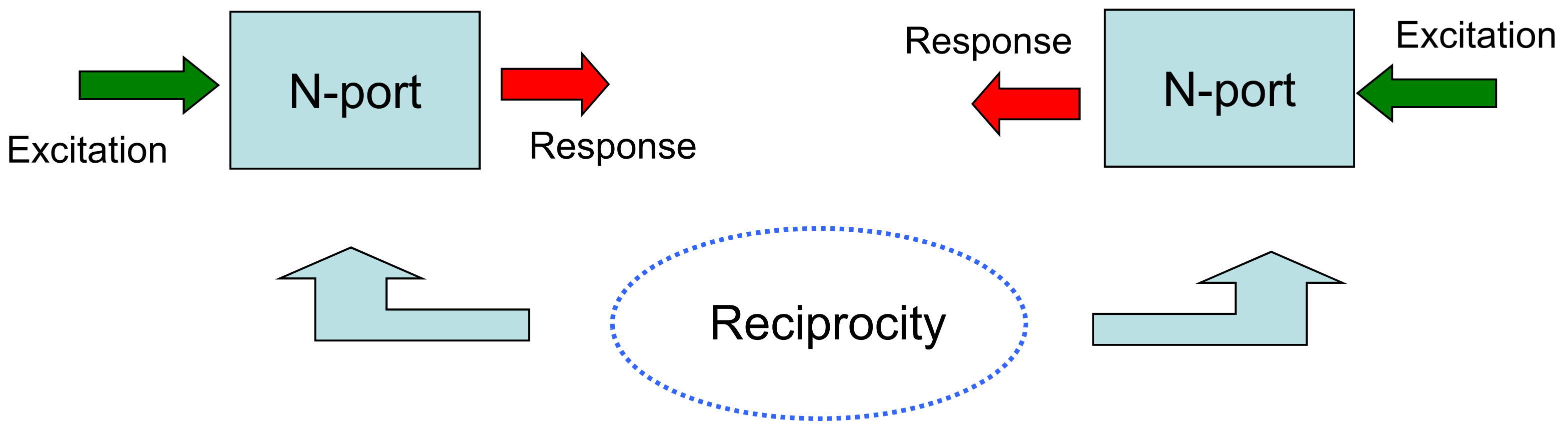
Independent (impressed) parameters**: Voltages at the ports**

$y_{i,i}$= Admittance at port *i* with all the others short circuited

$y_{i,j}$= Ratio of the current at port *i* e the impressed voltage at port *j* with all the ports short circuited.

Properties of Z and Y

1. Reciprocity. Response/Excitation remains the same when the ports are exchanged



For a reciprocal N-port network it has:

$$z_{i,j}=z_{j,i} \qquad y_{i,j}=y_{j,i}$$

i.e. **Z** and **Y** are symmetric

Properties of Z and Y

2. <u>Passivity</u>: Assuming no sources inside the network, in a passive network the sum of the power flowing through all the ports must be positive:

$$P = \frac{1}{2}\mathrm{Re}\left(V_1 I_1^* + V_2 I_2^* + \ldots V_N I_N^*\right) \geq 0$$

Note that the sign of V_i and I_i must be taken into account (the sign is positive for a power going into the port).
In matrix notation, the above equation becomes:

$$\mathbf{Q_Z} = \mathbf{Z} + \tilde{\mathbf{Z}}^* = \textit{Positive Defined}, \quad \mathbf{Q}_Y = \mathbf{Y} + \tilde{\mathbf{Y}}^* = \textit{Positive Defined}$$

In a *positive defined* matrix the determinant, the eigenvalues and the main diagonal are real and positive.

Properties of Z and Y

3. Lossless: in case of no dissipation inside the network, the overall power at the ports must be 0, then $\mathbf{Q}_Z=\mathbf{Q}_Y=0$. This determines (for **Z**):

$$z_{i,j} = -\left(z_{j,i}\right)^*$$

If the N-port network is also reciprocal ($z_{i,j}=z_{j,i}$) it turns out that all the element of **Z** must be imaginary.

Note that in case the reciprocity is not verified, the elements of **Z** on the main diagonal must be imaginary, while for the other it has ($z_{i,j}=r_{i,j}+jx_{i,j}$):

$$r_{i,j} = -r_{j,i}, \quad x_{i,j} = x_{j,i}$$

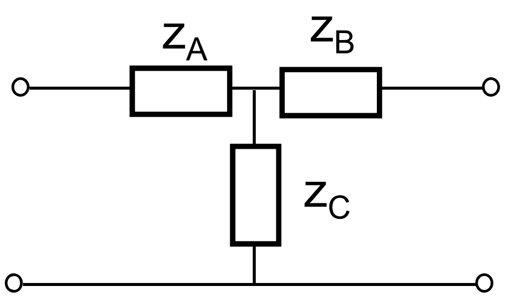
The same holds for **Y**

2-port network

For a lossless network Z and Y are imaginary. If it is also reciprocal:

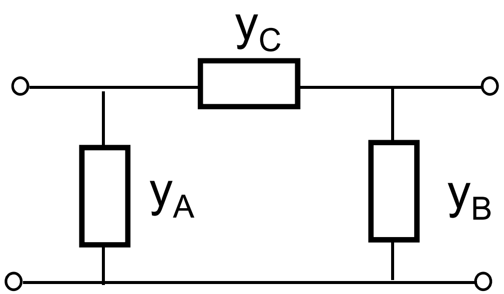
$$z_{12}=z_{21},\ y_{21}=y_{12}$$

Equivalent representation:



$$z_{11}=z_A+z_C,\ z_{22}=z_B+z_C,\ z_{12}=z_{21}=z_C$$

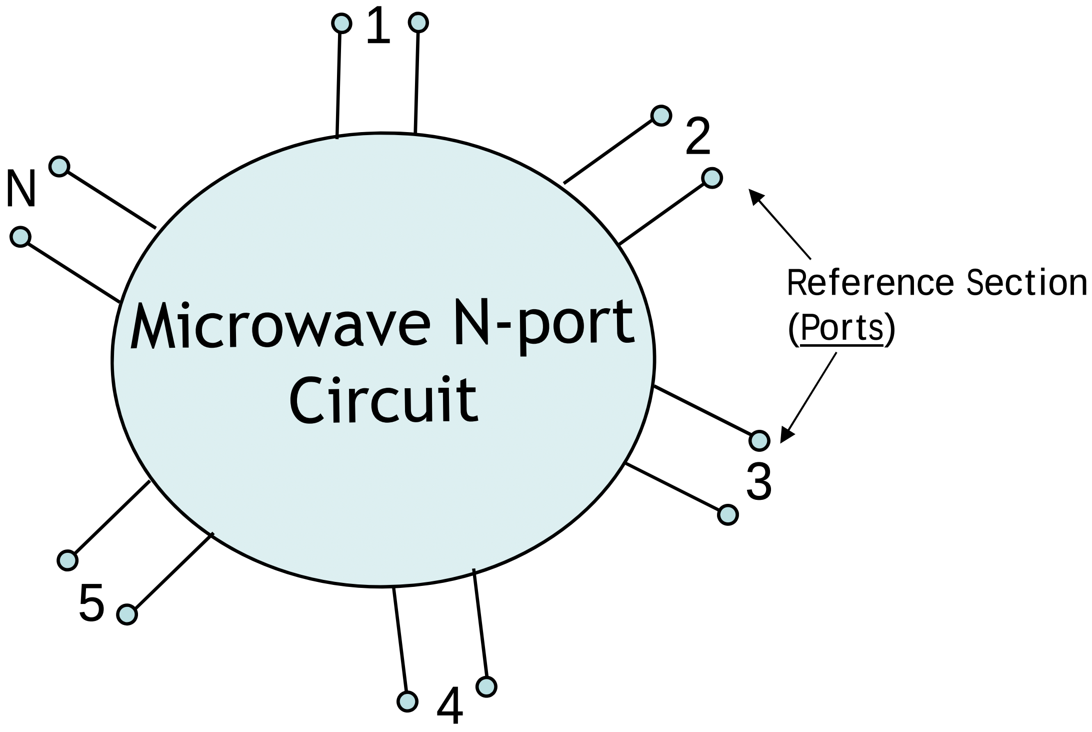
$$z_A=z_{11}-z_{12},\ z_B=z_{22}-z_{12},\ z_C=z_{12},$$



$$y_{11}=y_A+y_C,\ y_{22}=y_B+y_C,\ y_{12}=y_{21}=-y_C$$

$$y_A=y_{11}+y_{12},\ y_B=y_{22}+y_{12},\ y_C=-y_{12},$$

Microwave Linear Circuits



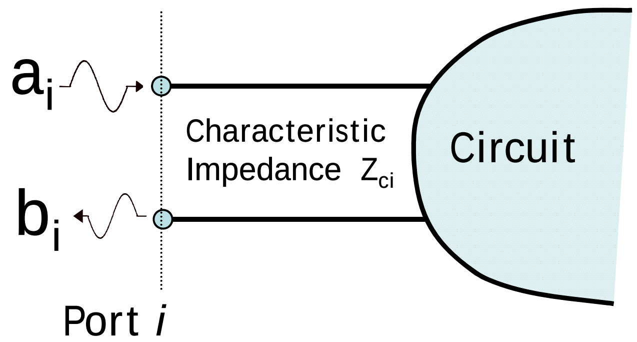
A microwave linear circuit is constituted by the interconnection of lumped and distributed elements. The interaction with the outer is realized through the external ports, which are defined on transmission lines outgoing from the network.

Note that the each port is characterized by a reference section, representing the position on the line where is collocated

Drawbacks of matrices Z e Y

- The direct measurement of the elements of Z and Y would require open or a short circuits, which may be difficult to realize at microwave frequencies
- The placement of a short or an open at the ports of active circuits may damage the active components
- The change of the reference sections requires complex transformations when Z or Y matrices are used
- Voltages and Currents are not always univocally defined in microwave circuits (it happens for TEM modes only)

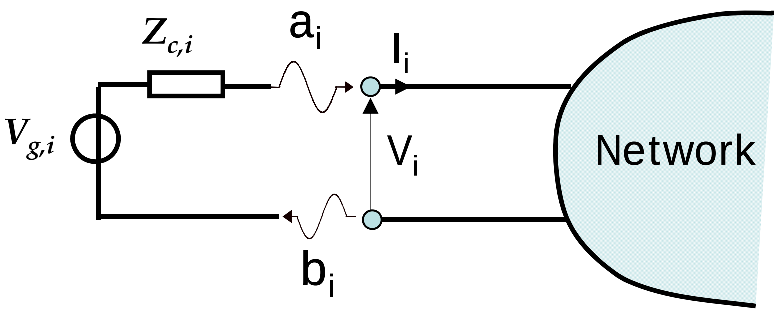
Power Wave definition



(1/2)|a_i|^2: Incident power wave = Available power from a source with impedance Z_{ci}

(1/2)|b_i|^2: Reflected power wave = Difference between the available power and the power absorbed by the port (i.e. flowing into the port)

Definition of conventional *V* and *I*



$$V_{g,i} = Z_{c,i} I_i + V_i$$

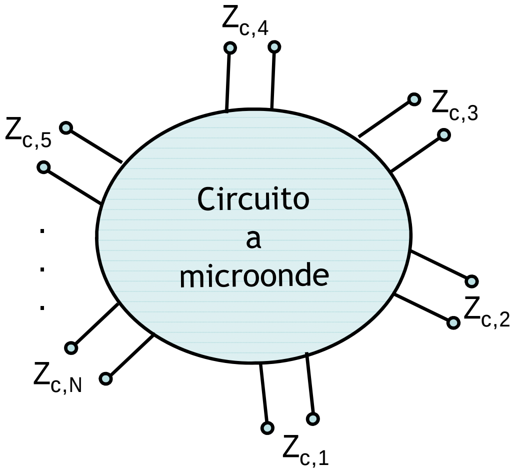
$$P_{IN,i} = \frac{1}{2} \mathrm{Re}\left\{ V_i \cdot I_i^* \right\}$$

$$P_{av,i} = \frac{1}{2}\left|a_i\right|^2 = \frac{\left|V_{g,i}\right|^2}{8 \cdot \mathrm{Re}\left\{Z_{c,i}\right\}} = \frac{\left|V_i + Z_{c,i} \cdot I_i\right|^2}{8 \cdot \mathrm{Re}\left\{Z_{c,i}\right\}} \quad \Rightarrow \quad a_i = \frac{V_i + Z_{c,i} \cdot I_i}{2\sqrt{\mathrm{Re}\left\{Z_{c,i}\right\}}} = I_i \frac{Z_i + Z_{c,i}}{2\sqrt{\mathrm{Re}\left\{Z_{c,i}\right\}}}$$

$$P_{IN,i} = \frac{1}{2} \mathrm{Re}\left\{V_i \cdot I_i^*\right\} = \frac{1}{2}\left(\left|a_i\right|^2 - \left|b_i\right|^2\right) \quad \Rightarrow \quad b_i = \frac{V_i - Z_{c,i}^* \cdot I_i}{2\sqrt{\mathrm{Re}\left\{Z_{c,i}\right\}}} = I_i \frac{Z_i - Z_{c,i}^*}{2\sqrt{\mathrm{Re}\left\{Z_{c,i}\right\}}}$$

$$\left(a_i = \frac{I_i + Y_{c,i} \cdot V_i}{2\sqrt{\mathrm{Re}\left\{Y_{c,i}\right\}}}, \quad b_i = -\frac{I_i + Y_{c,i}^* \cdot V_i}{2\sqrt{\mathrm{Re}\left\{Y_{c,i}\right\}}} \right)$$

Generalized Scattering Matrix



For a linear circuit, incident and reflected waves are linearly related:

$$
\begin{aligned}
b_1 &= s_{11}a_1 + s_{12}a_2 + ... + s_{1N}a_N \\
b_2 &= s_{21}a_1 + s_{22}a_2 + ... + s_{2N}a_N \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
b_N &= s_{N1}a_1 + s_{N2}a_2 + ... + s_{NN}a_N
\end{aligned}
$$

In matrix form:

$$\underline{\mathbf{b}} = \underline{\underline{\mathbf{S}}} \cdot \underline{\mathbf{a}} \qquad \underline{\underline{\mathbf{S}}} = \begin{pmatrix} s_{11} & \cdots & s_{1N} \\ \vdots & \ddots & \vdots \\ s_{N1} & \cdots & s_{NN} \end{pmatrix}$$

Meaning of S parameters

$$s_{ii} = \left.\frac{b_i}{a_i}\right|_{a_{k \neq i}=0}$$

Reflection coefficient at port i when the other ports are connected to their reference impedances Z_{cj} (*matched*)

$$s_{ij} = \left.\frac{b_i}{a_j}\right|_{a_{k \neq j}=0}$$

Transmission coefficient from port j to port i with the other ports matched. Note that $|s_{ij}|^2$ rappresents the transducer power gain between the two ports

Properties of S Matrix

- For a reciprocal network **S** is symmetric ($s_{ij}=s_{ji}$)
- For a lossless network **S** is unitary ($\mathbf{S}\cdot\widetilde{\mathbf{S}^{*}}=\mathbf{U}$)
- If the reference sections are shifted by d_i the new matrix S' is given by:
  $\mathbf{S}'=\mathbf{\Phi}\cdot\mathbf{S}\cdot\mathbf{\Phi}$ where Φ is diagonal matrix with elements $\exp(j\beta d_i)$
  It has: $s'_{i,j}=s_{i,j}\cdot e^{-j\beta\left(d_i+d_j\right)}$
  Note that d_i is positive with an outward shift of the section
- If the reference impedances at the ports are changed from $Z_{c,i}$ to $Z'_{c,i}$ the new matrix S' is given by the following equation:

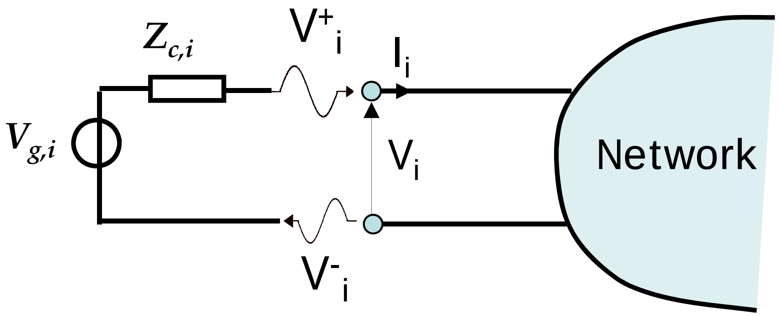
$$\mathbf{S}'=\mathbf{A}^{-1}\cdot\left(\mathbf{S}-\tilde{\mathbf{\Gamma}}^{*}\right)\cdot\left(\mathbf{U}-\mathbf{\Gamma}\cdot\mathbf{S}\right)^{-1}\tilde{\mathbf{A}}^{*}$$

with $\mathbf{\Gamma}$ and $\mathbf{A}$ diagonal matrices given by:

$$\mathbf{\Gamma}=\left(\mathbf{Z}'-\mathbf{Z}\right)\cdot\left(\mathbf{Z}'+\tilde{\mathbf{Z}}^{*}\right)^{-1},\quad \mathbf{A}=diag\left\{A_{ii}\right\},\quad A_{ii}=\left|1-\Gamma_{ii}\right|^{-1}\left(1-\Gamma^{*}_{ii}\right)\left(1-\left|\Gamma_{ii}\right|^{2}\right)^{1/2}$$

$$\mathbf{Z}=diag\left\{Z_{c,i}\right\},\quad \mathbf{Z}'=diag\left\{Z'_{c,i}\right\}$$

Power and Voltages waves



$$V_i = V_i^+ + V_i^-, \quad I_i = I_i^+ + I_i^-$$

$$V_i^+ = Z_{c,i} I_i^+, \qquad V_i^- = -Z_{c,i} I_i^-$$

Power waves:

$$a_i = \frac{V_i + Z_{c,i} \cdot I_i}{2\sqrt{\mathrm{Re}\{Z_{c,i}\}}}, \quad b_i = \frac{V_i - Z_{c,i}^* \cdot I_i}{2\sqrt{\mathrm{Re}\{Z_{c,i}\}}}$$

Voltage waves:

$$V_i^+ = \frac{V_i + Z_{c,i} \cdot I_i}{2}, \quad V_i^- = \frac{V_i - Z_{c,i} \cdot I_i}{2}$$

Absence of reflected wave:

Conjugate matching

$$Z_i = Z_{c,i}^*$$

Matching

$$Z_i = Z_{c,i}$$

Coincide for $Z_{c,i}$ real

Particular case: real $Z_{c,i}$

With $Z_{c,i}$ ($Y_{c,i}$) real, the power and voltage waves coincide (unless a constant).
The relationship between S and Z (or Y) simplifies to:

$$\mathbf{S} = \left(\mathbf{Z_N} - \mathbf{U}\right)\cdot\left(\mathbf{Z} + \mathbf{U}\right)^{-1}, \qquad \mathbf{Z_N} = \left(\mathbf{U} - \mathbf{S}\right)^{-1}\cdot\left(\mathbf{S} + \mathbf{U}\right)$$

$$\mathbf{S} = \left(\mathbf{U} - \mathbf{Y_N}\right)\cdot\left(\mathbf{U} + \mathbf{Y_N}\right)^{-1}, \qquad \mathbf{Y_N} = \left(\mathbf{U} + \mathbf{S}\right)^{-1}\cdot\left(\mathbf{U} - \mathbf{S}\right)$$

where $\mathbf{Z_N}$ e $\mathbf{Y_N}$ are normalized matrices defined as:

$$\mathbf{Z_N} = \left[\frac{z_{i,j}}{\sqrt{Z_{c,i}\cdot Z_{c,j}}}\right], \qquad \mathbf{Y_N} = \left[\frac{y_{i,j}}{\sqrt{Y_{c,i}\cdot Y_{c,j}}}\right]$$

Note that if $Z_{c,i}=Z_0$ (equal for all the ports): $Z_N(i,j)=Z(j,i)/Z_0$