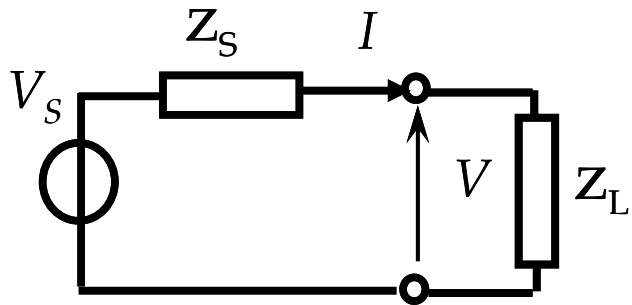
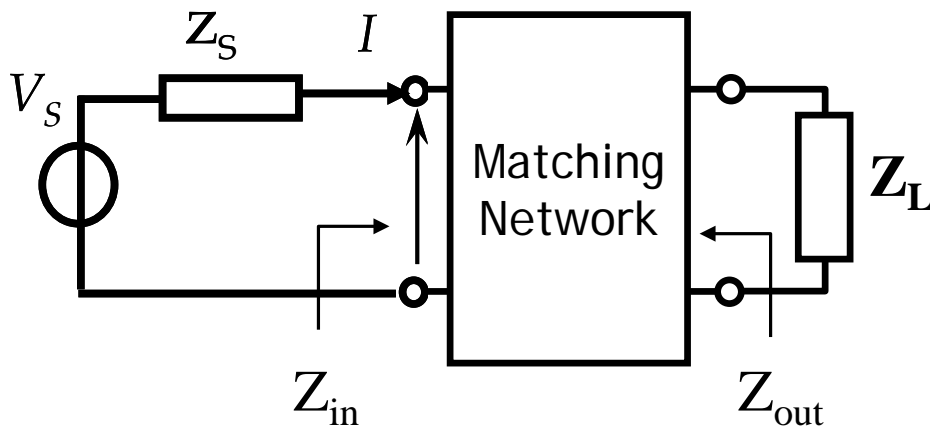


# Available power from a source and conjugate matching



The Available power of a source is the maximum power the source can deliver to a load.

$$P_{av} = \frac{1}{8} \frac{|V_S|^2}{\text{Re}(Z_S)} \quad Z_L \neq Z_S^* \Rightarrow P_L < P_{av}$$



$$Z_{in} = Z_S^* \quad \text{Conjugate Matching}$$

$$Z_{out} = Z_L^*$$

Lossless Network



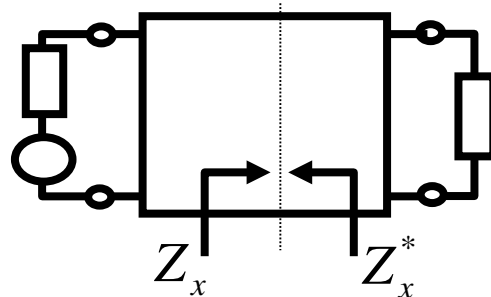
$$P_L = P_{av}$$

# Matching and conjugate matching

Matching ( $Z_{in}=Z_g$ ) and conjugate matching are in general different. They coincide only for real source (load) impedance. For instance when the source is connected to the load with a transmission line of characteristic impedance  $Z_c$  (real), all the available power is transferred to the load if  $Z_L=Z_g=Z_c$

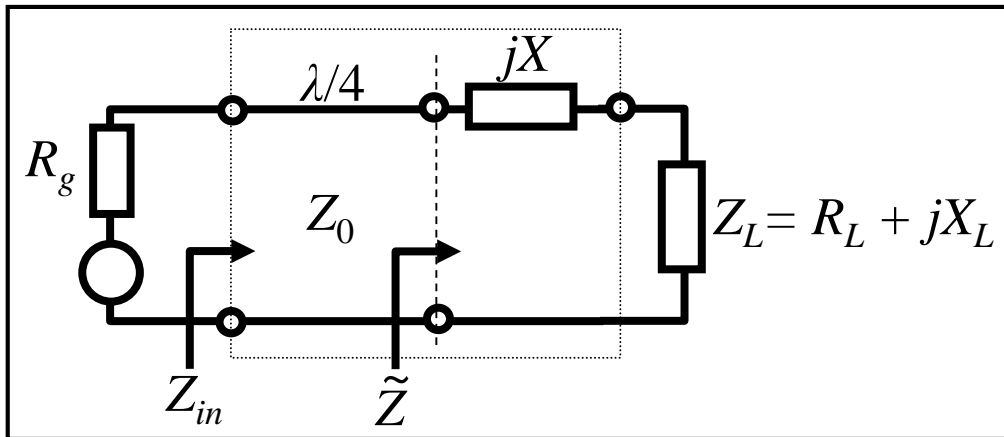
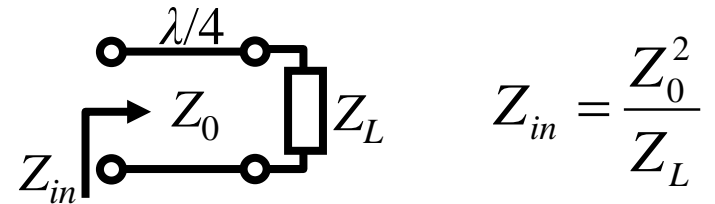
**Property:** For lossless matching network ( $\rightarrow$  only not dissipative components), if the conjugate matching is verified at one section (included input or output), it must be verified in any other section.

As a consequence it is sufficient to impose matching condition only at one section (typically at input)



# Matching with $\lambda/4$ line

**Note:** a line with  $\lambda/4$  length represents an **impedance inverter**:



$$Z_{in} = \frac{Z_0^2}{R_L + j(X_L + X)} = Z_g^* = R_g$$

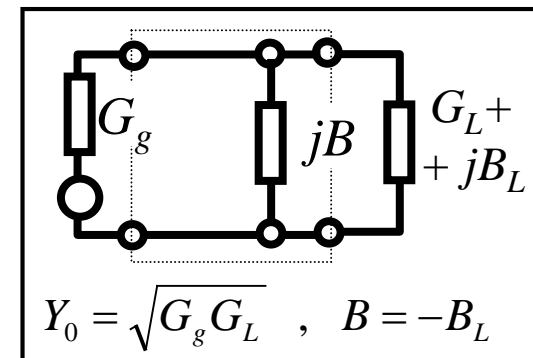
$$\Downarrow$$

$$X = -X_L$$

$$Z_0 = \sqrt{R_g R_L}$$

Note:

- The same concept can be applied with reference to the admittance.
- The reactance (or susceptance) can be realized with lumped or distributed (stubs) components.



# Single Stub Matching

Stubs (s.c. or o.c.) can be employed for realizing the reactances (susceptances).

**Assumption:** Source matched to the line ( $G_G = Y_c$ )

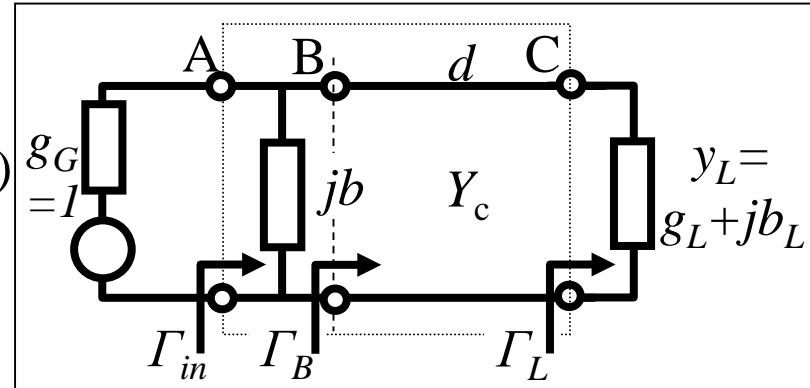
$$\downarrow$$

$$\text{Matching: } \Gamma_{in} = 0 \quad (\rightarrow y_{in} = 1)$$

Sequence of operations:

- Displacement at  $\Gamma = \text{const.}$  (line  $d$ )
- Displacement at  $g = \text{const.}$  (susceptance  $b$ )

$$\hookrightarrow y_{in} = g_B + j(b_B + b) = g_G = 1 \quad \rightarrow \underline{\Gamma_B \text{ must be on the circle } g = 1}$$



## Procedure

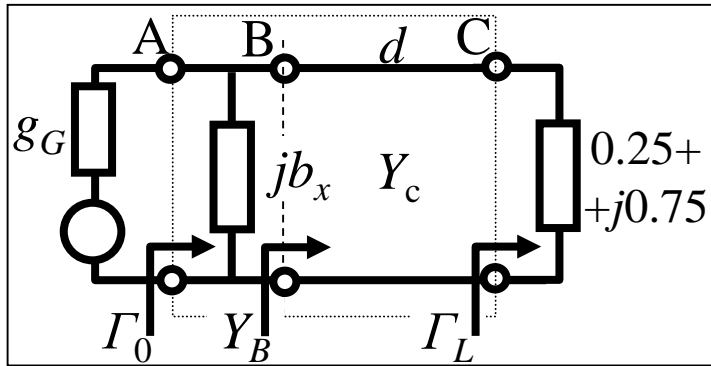
- Evaluation of  $d$  for transforming  $\Gamma_L$  into  $\Gamma_B$  on the circle  $g = 1$ .
- Evaluation of  $b$  for transforming  $\Gamma_B$  into  $\Gamma_{in} = 0$
- Dimensioning of the stub implementing the susceptance  $b$ .

## Note

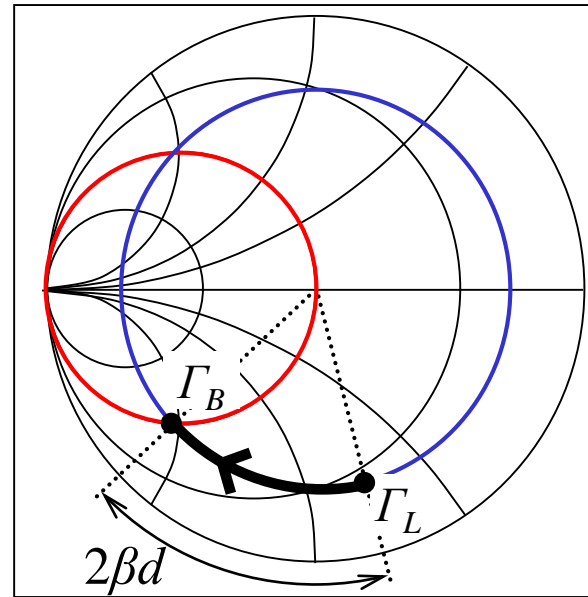
- It is possible to design with the same procedure the dual network employing a series-connected stub.

# Single Stub - Example

$$Z_c = 50 \Omega; Z_G = Z_c; \epsilon_r = 4; f_0 = 3 \text{ GHz}; y_L = Y_L/Y_c = 0.25 + j0.75j$$



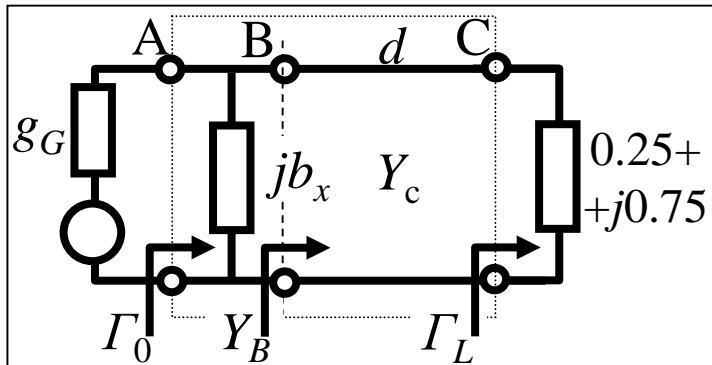
- Circle  $|\Gamma| = |\Gamma_L|$
- Circle  $g = g_G = 1$



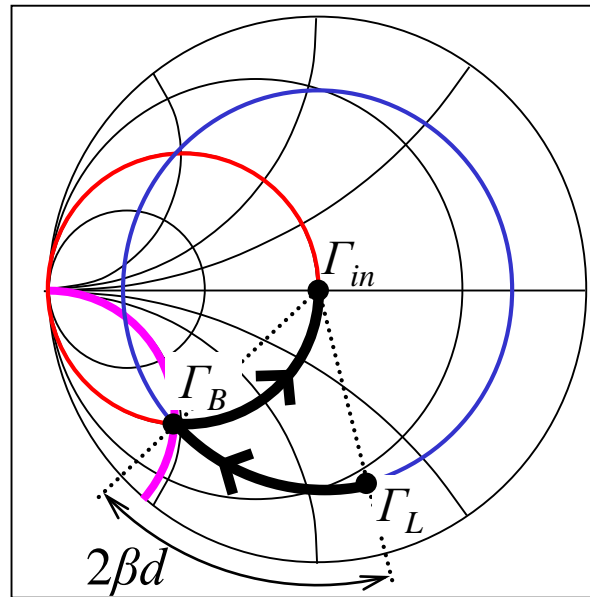
- Enter  $y_L$  on S.C.:  $\Gamma_L = 0.728 \angle -76^\circ$

- Evaluation of  $d$ :  $2\beta d \cong 60.7^\circ \cong 0.337\pi \Rightarrow d = \frac{0.337\pi}{2 \cdot 2\pi f_0 \sqrt{\epsilon_r} / c} = 4.2 \text{ mm}$

# Single Stub - Example 2



- Circle  $|\Gamma| = |\Gamma_L|$
- Circle  $g = g_G$
- Circle  $b = b_B$

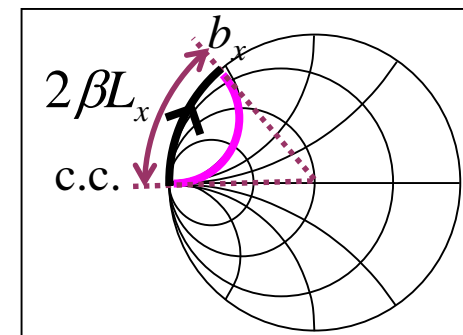


- Evaluation of  $b_x$ :

$$Y_B = 1 + j2.12$$

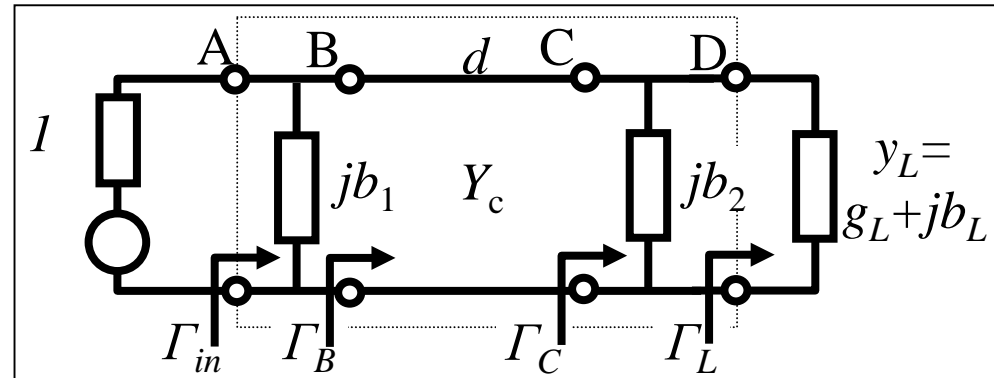
$$y_0 = y_B + jb_x = 1 + j(b_x + 2.12) = 1 \rightarrow b_x = -2.12$$

- Dimensioning of the stub in short circuit:  $2\beta L_x \cong 0.28\pi \Rightarrow L_x \cong 3.5 \text{ mm}$



# Double Stub Matching

Two stubs separated by a transmission line of assigned length  $d$  are employed.



Source matched to the line  $\rightarrow \Gamma_{in} = 0$  ( $\rightarrow y_{in} = 1$ )

**Note:** Sequence of displacements:

- Displacement at  $g = \text{const.}$  (susceptance  $b_2$ )
- Displacement at  $\Gamma = \text{const.}$  (line  $d$ )
- Displacement at  $g = \text{const.}$  (susceptance  $b_1$ )

$$y_{in} = g_B + j(b_B + b_1) = 1 \rightarrow g_B = 1, (b_B + b_1) = 0 \rightarrow \Gamma_B \text{ must be on the circle } g = 1$$



The displacement with  $d = \text{const}$  must transform  $\Gamma_C$  into a point on the circle  $g = 1$



$\Gamma_L$  is transformed into  $\Gamma_C$  through a displacement on the circle  $g = g_L$

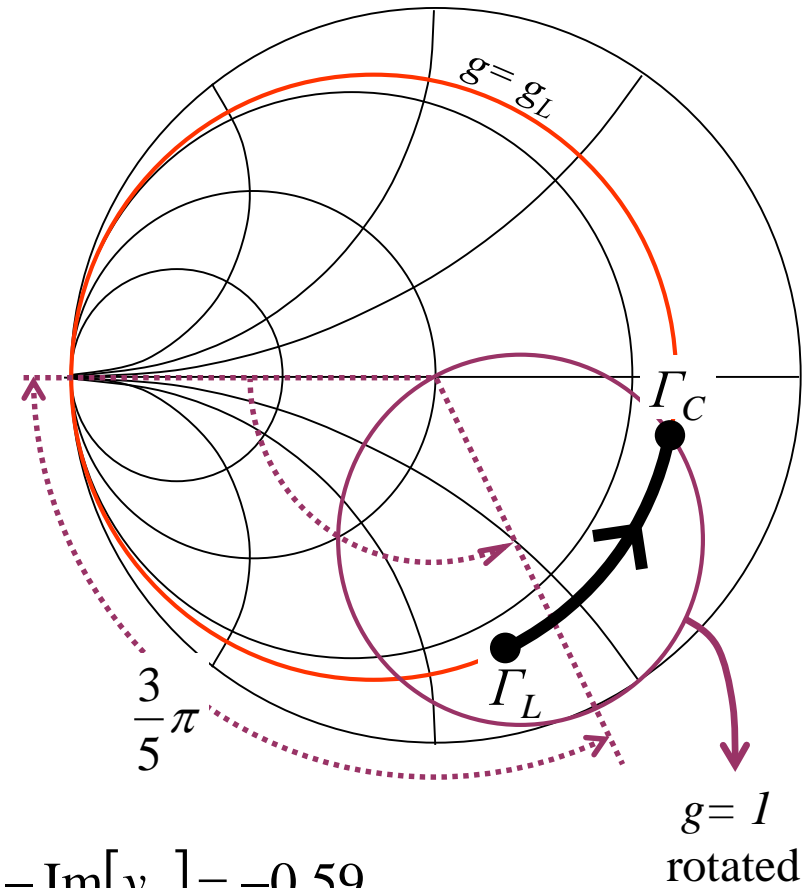
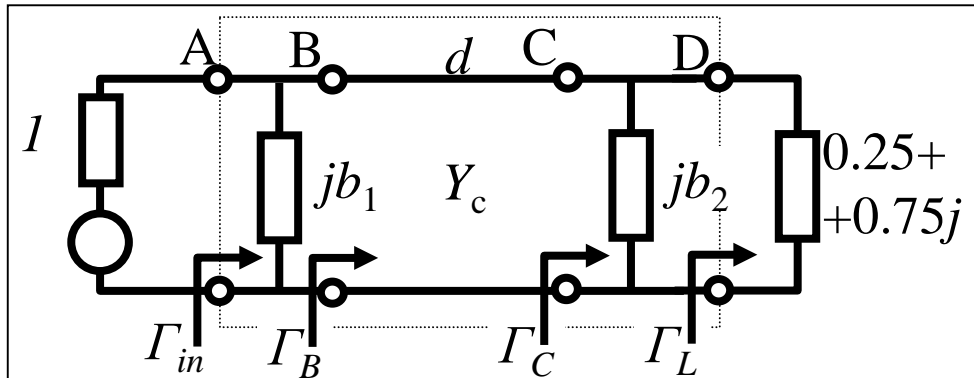




# Double Stub - Example 1

$$Z_c = 50 \Omega; Z_G = Y_c; \epsilon_r = 4; f_0 = 3 \text{ GHz}; y_L = Y_L/Y_c = 0.25 + 0.75j$$

$$d = 7.5 \text{ mm}$$

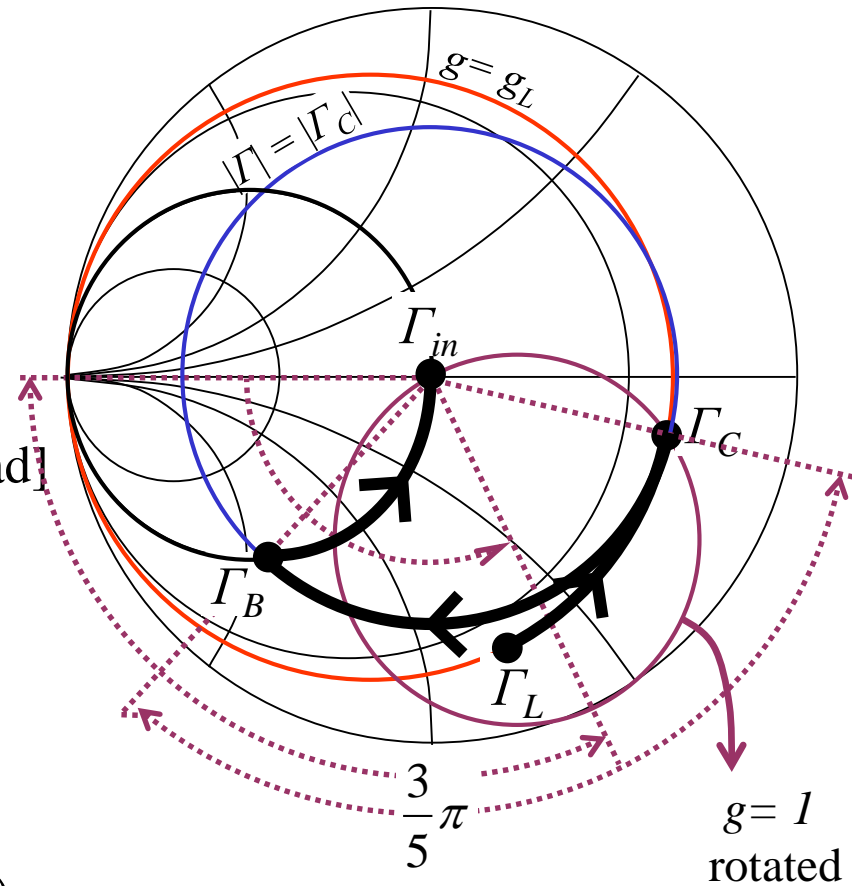
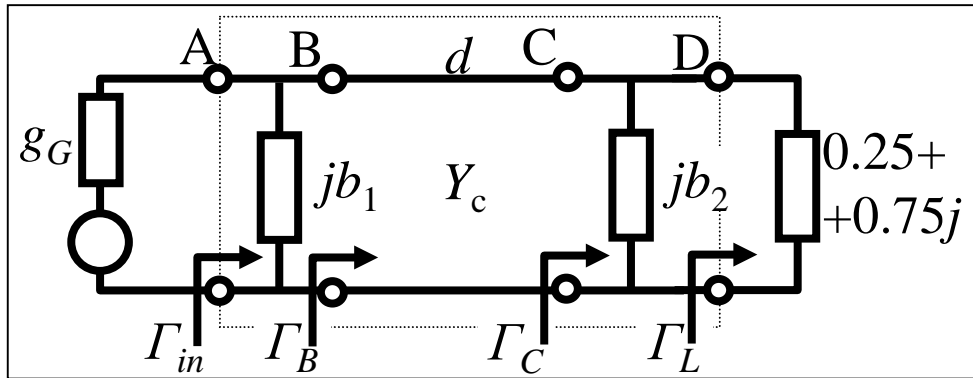


- Circles  $g = g_L$  e  $g = 1$  rotated  $\rightarrow 2\beta d = 2 \cdot \frac{2\pi f_0}{c/\sqrt{\epsilon_r}} d = \frac{3}{5}\pi$

$$\Gamma_C = 0.61 \angle -19.8^\circ, \quad y_C = 0.25 + j \overbrace{0.164}^{b_C}$$

- Evaluation of di  $b_2$ :  $b_2 = b_C - b_L = \text{Im}[y_C] - \text{Im}[y_L] = -0.59$

# Double Stub - Example 2



- Circle  $|\Gamma| = |\Gamma_C|$  and negative rotation of  $2\beta d$  [rad]

$$\Gamma_B = 0.61 \angle -127.8^\circ, \quad y_B = 1 + j \overbrace{1.540}^{b_B}$$

- Evaluation of  $b_1$ :  $b_1 = -b_B = -1.540$

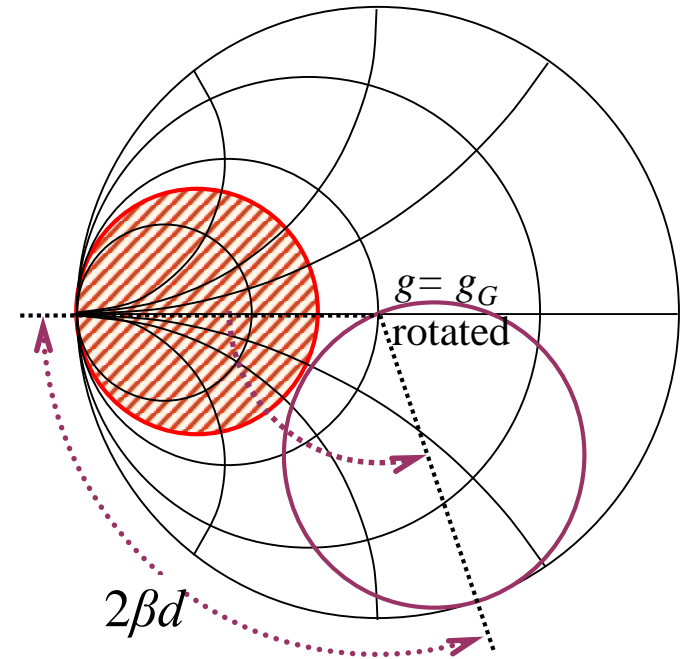
- Stub  $b_1$  in s.c.:  $\varphi_1 = \beta L_1 = \tan^{-1}(-b_1^{-1}) = 0.58 \text{ rad} \Rightarrow L_1 = 4.6 \text{ mm}$

- Stub  $b_2$  in s.c.:  $\varphi_2 = \beta L_2 = \tan^{-1}(-b_2^{-1}) = 1.04 \text{ rad} \Rightarrow L_2 = 8.3 \text{ mm}$

# Double Stub - Notes

- For a given value of  $d$ , there is a *forbidden region* which represents the values of  $\Gamma_L$  that cannot be matched by the network.

In this region, the circles  $g = g_L$  and  $g = 1$  rotated do not intersect. With  $d = 3/8 \lambda_0$  the forbidden region is in the interior of the circle  $g=2$ .



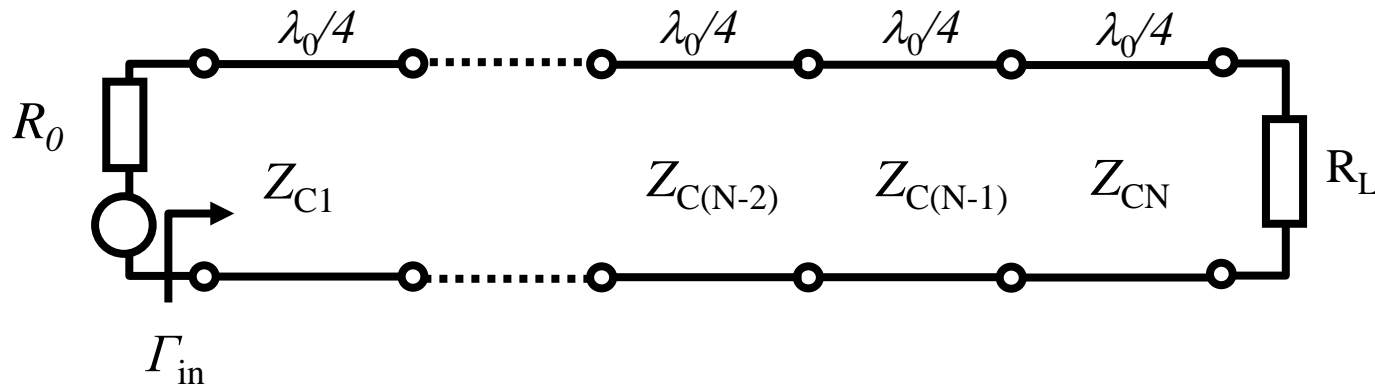
- It is possible also the dual topology, with the stubs connected in series .

# Frequency dependence of matching

- It is advisable to verify how  $|\Gamma_{in}|$  changes with frequency (the matching specs are generally given in a band centered at  $f_0$ )
- However, matching depends on the relative bandwidth  $B_n (=B/f_0)$ . At microwave frequencies, small values of  $B_n$  (few %), correspond to relevant values of absolute bandwidth
- For this reason, in case of narrow band (less than 1-2%), it is not necessary to take into account the frequency dependence
- On the other hand, when the  $B_n$  is not small or the matching level ( $|\Gamma_{in}|$ ) in the band is very small, the design of the matching network becomes a not easy task. In practice, being analytical techniques not available nor convenient, the design is based on numerical optimization of the components parameters values
- In general, an increase of the number of components in the matching network is requested with the increase of the bandwidth (for a given matching level). However, the improvement of matching gets smaller and smaller with the increase of the components number

# Broadband matching: a special case

When source and load to match are both resistive, it is possible to broaden the matching bandwidth using several cascaded sections of transmission lines with length  $\lambda_0/4$  (*commensurate transformer*):



Imposing that the first  $N-1$  derivatives of  $|\Gamma_{in}(\omega)|$  vanish at  $f_0$  we have:

$$\ln \frac{Z_{C(n+1)}}{Z_{C(n)}} \cong 2^{-N} \frac{N!}{(N-n)!n!} \ln \frac{R_L}{R_0} \quad (\text{per } 0.5R_0 < R_L < 2R_0)$$

For dimensioning the network, it is first selected the number of sections to be cascaded ( $N$ ). Then assigning  $n=0$  in the above equation  $Z_{C1}$  is evaluated ( $Z_{C(0)}=R_0$ ). The other unknowns ( $Z_{C(k)}$ ) are obtained by applying recursively the above expression.

# Example

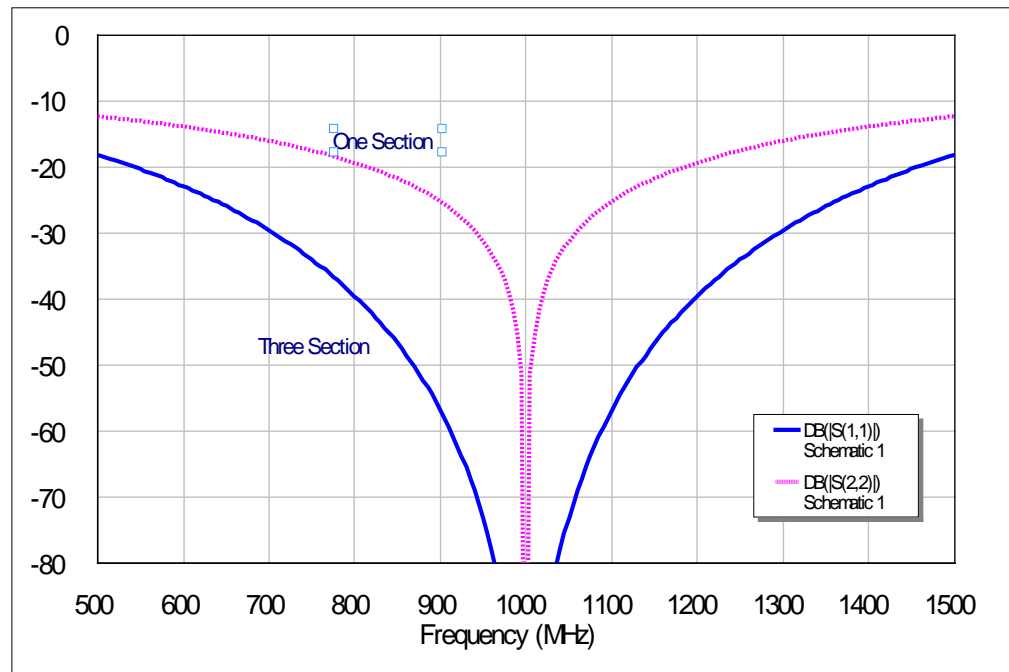
We have:  $R_L=100 \Omega$ ,  $R_0=50\Omega$ . With  $N=3$  the following results are obtained:

$$\ln \frac{Z_{C1}}{R_0} \cong \frac{6}{8.6} \ln \frac{R_L}{R_0} \Rightarrow Z_{C1} = 54.525$$

$$\ln \frac{Z_{C2}}{Z_{C1}} \cong \frac{3}{8} \ln \frac{R_L}{R_0} \Rightarrow Z_{C2} = 70.7107$$

$$\ln \frac{Z_{C3}}{Z_{C2}} \cong \frac{3}{8} \ln \frac{R_L}{R_0} \Rightarrow Z_{C3} = 91.7$$

$|\Gamma_{in}|$  computed with a  
CAD software:



# Matching of Antennas

- ❑ In the operation band the real part of the impedance seen at the electrical port of an antenna (radiation impedance) is about equal to the reference impedance ( $50 \Omega$ )
- ❑ The imaginary part of  $Z$  may be different from 0 even at the center frequency and then it causes a certain degree of mismatch in the operation band of the antenna
- ❑ The use of a matching network then allows to improve noticeably the radiation efficiency in the operative antenna band
- ❑ The simplest network is represented by a single stub in series (or parallel) to the antenna port, dimensioned for zeroing the imaginary part of  $Z$  (or  $Y$ ) in the center frequency
- ❑ Analytical procedures for the design can be used only in the vary simple cases (single frequency matching). Numerical optimization is generally used to take into account the frequency behavior of the matching

# Optimization of electrical network

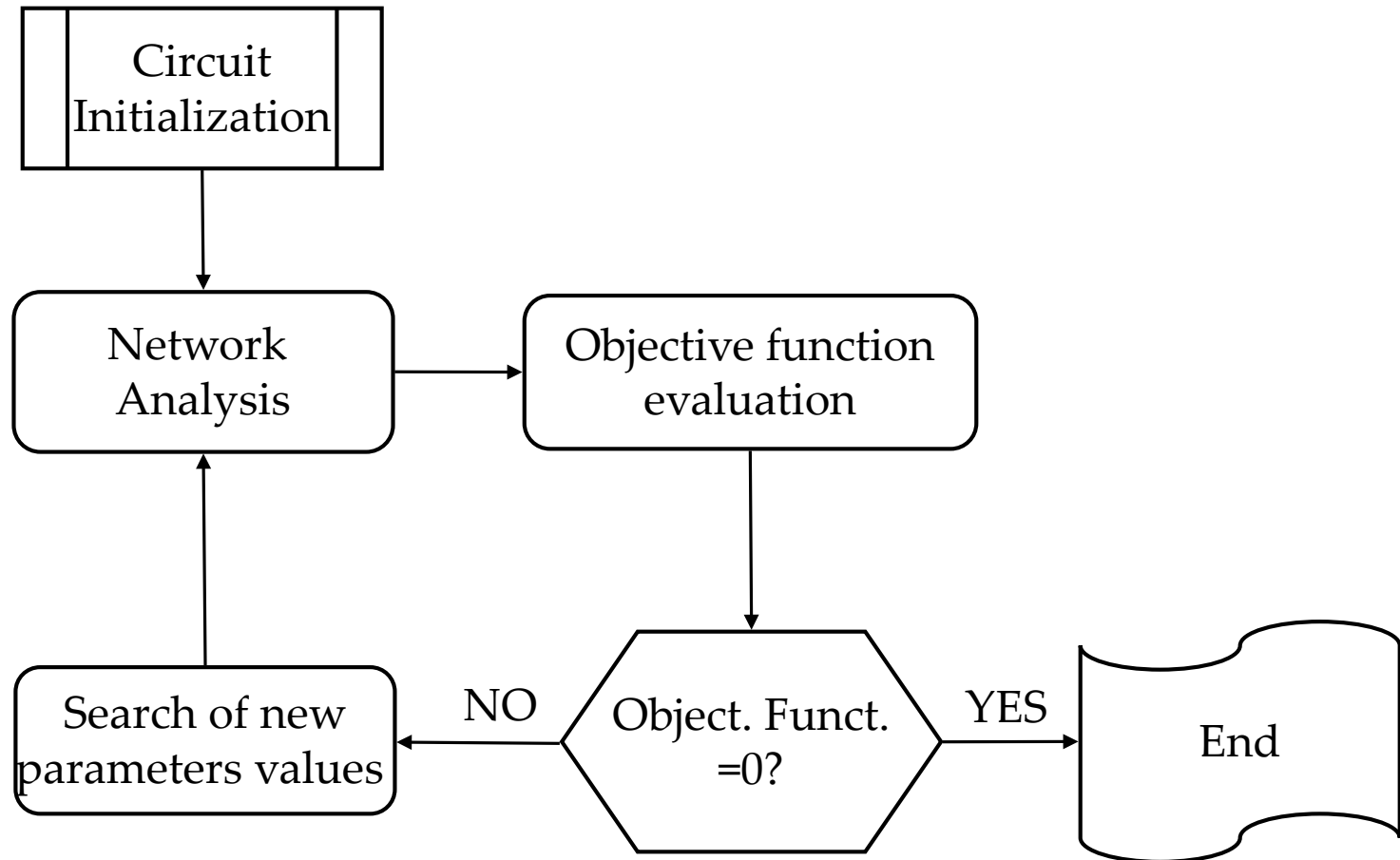
**Goal:** Given a network with specific topology, find the values of the components parameters so that the network response meets one (or more) desired objectives

## Definitions

- *Objective function:* mathematical representation of the desired objectives. The closer is the response of the network to the objectives , the smaller is the value of the objective function
- *Search Algorithm:* define the search procedure (typically iterative) of the optimum result (= the parameters values for which the objective function is equal to zero)



# General Scheme



# Objective Function

Network Function:  $H(f_i, p_j)$

$f_i$ :  $f_1, f_2, \dots, f_M$  = Frequency analysis

$p_j$ :  $p_1, p_2, \dots, p_N$  = Network parameters

The frequency points must be defined at the beginning, together with the corresponding values that the network function must present ( $H_d(f_i)$ ).

$p_1 \dots p_n$  represent the network parameters to be optimized (they must be initialized at the beginning of the procedure)

Least-Square function: 
$$OF(\mathbf{p}) = \frac{1}{M} \sum_{i=1}^M |H_d(f_i) - H(f_i, \mathbf{p})|^2$$

Least-q function : 
$$OF(\mathbf{p}) = \frac{1}{M} \sum_{i=1}^M |H_d(f_i) - H(f_i, \mathbf{p})|^{2q} \quad (q \text{ integer } > 1)$$

Min-Max function : 
$$OF(\mathbf{p}) = \underset{i=1, \dots, M}{Max} \left\{ |H_d(f_i) - H(f_i, \mathbf{p})| \right\}$$

# Multiple Objective Functions

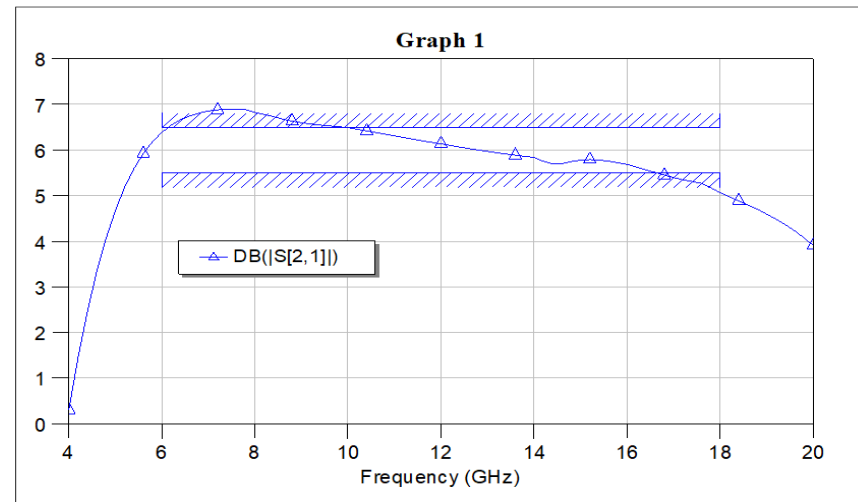
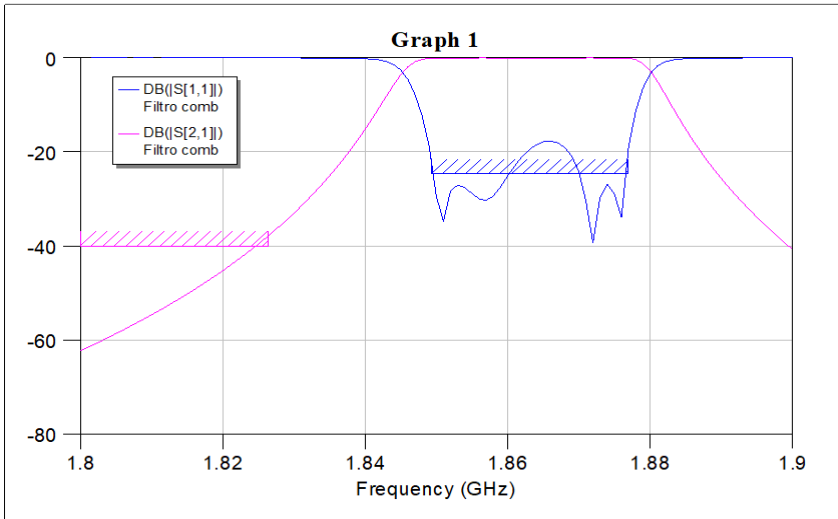
Different kinds of network functions:

$$FO(\mathbf{p}) = \frac{\alpha_1}{M_1} \sum_{i=1}^{M_1} |H_{d1}(f_i) - H_1(f_i, \mathbf{p})|^2 + \frac{\alpha_2}{M_2} \sum_{i=1}^{M_2} |H_{d2}(f_i) - H_2(f_i, \mathbf{p})|^2 + \dots$$

Different values depending on the network function value:

$$FO(\mathbf{p}) = \begin{cases} \frac{1}{M} \sum_{i=1}^M |H_{d1}(f_i) - H(f_i, \mathbf{p})|^2 & \text{for } H(f_i) > H_{d1}(f_i) \\ 0 & \text{for } H_{d1}(f_i) < H(f_i) < H_{d2}(f_i) \\ \frac{1}{M} \sum_{i=1}^M |H_{d2}(f_i) - H(f_i, \mathbf{p})|^2 & \text{for } H(f_i) < H_{d1}(f_i) \end{cases}$$

# Examples of multiple objective functions



# Optimization Algorithms

Define the procedure for the selection of the “next point” during the search of the optimum.

Note that the “optimum”, from a mathematical point of view, is represented by the absolute minimum of the objective function in the n-dimensional space of the unknown network parameters

## Classification

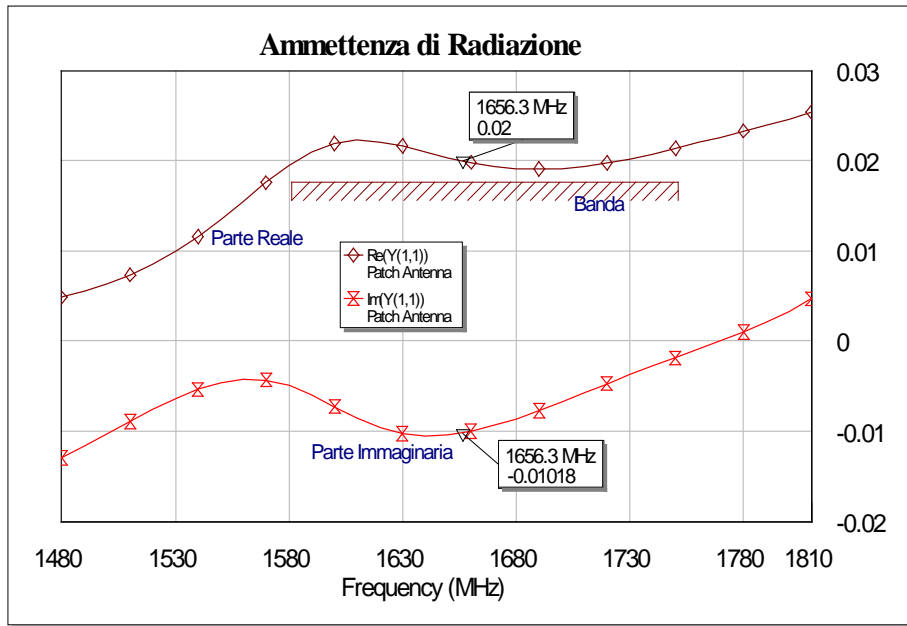
- ❑ *Random search*: the next point is obtained by attempts, by assigning randomly the values of the unknowns (the limits of the n-dimensional space must be however specified). A large number of OF evaluations is requested, but the search is little influenced by local minima of the OF.
- ❑ *Determinist Algorithms*: are based on the analytical search of the minimum of the OF in the n-dimensional space of the unknowns; can be classified in:
  - Derivative-based algorithms (Newton, Quasi-Newton, Gauss-Newton)
  - “pattern-search” algorithms, not requesting the use of the derivatives (simplex, genetic, simulated annealing)

They are in general more accurate and faster of the previous ones, but are very sensitive to local minima (so the final result is much dependent on the initial assignments of the unknowns )

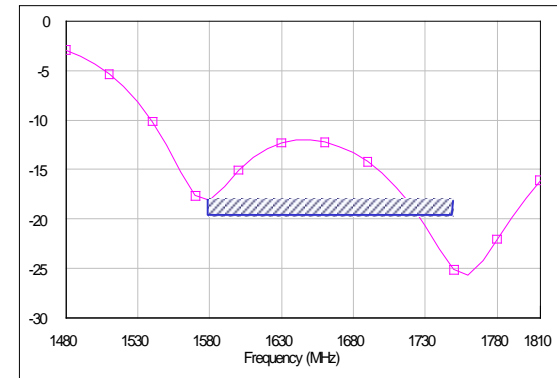
# Comments

- **Initialization:** the final result of the optimization procedure is in general strongly affected by the initial values assigned to the unknowns; it is always convenient to get analytically the starting point (for instance, through the synthesis at the center frequency)
- **“Divide and Conquer”:** The higher is the overall number of unknowns, the lower is the probability of finding the optimum. It is convenient, if possible, to subdivide the original (large) problem into a set of smaller sub-problems to be optimized. The solution of the sub-problems are then used as starting point for the optimization of the big problem
- **Algorithms:** the random-type algorithms are useful when the initial point is not enough good. They in fact make it more likely to skip the local minima, at expense of many evaluations of the OF. When the solution approaches a good region (small values of the OF) it is convenient to switch to a deterministic algorithm to increase the computation speed and accuracy

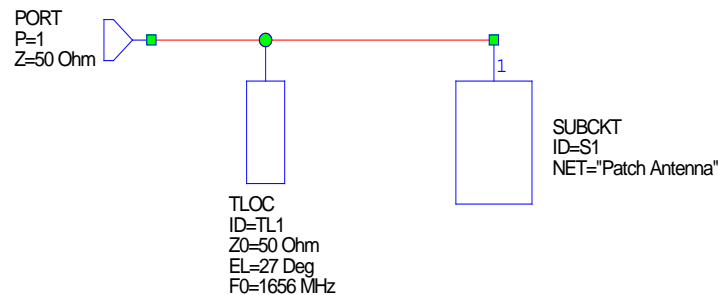
# Example: matching of a patch antenna



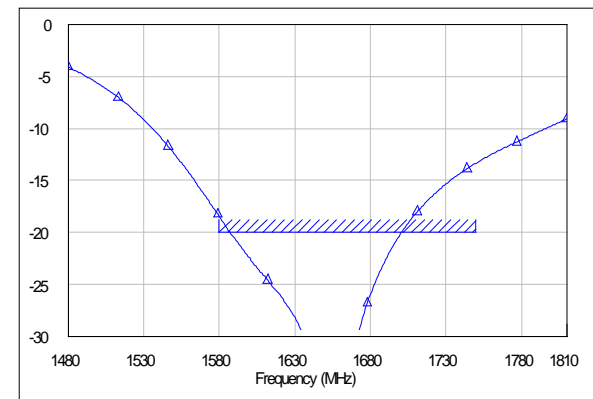
Operation band: 1580-1750 MHz  
In band Return Loss: > 20 dB



Without match

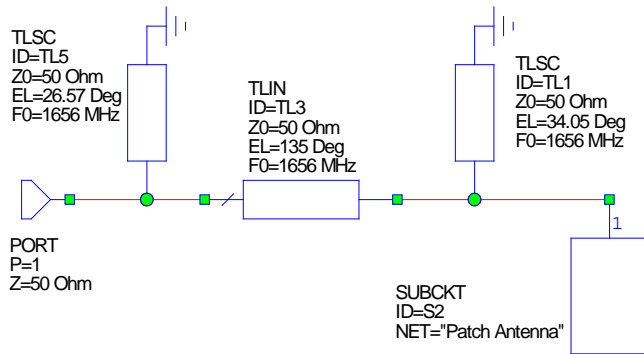


$$\beta L_s = \tan^{-1} \left( \frac{0.01018}{0.02} \right) = 26.98^\circ$$

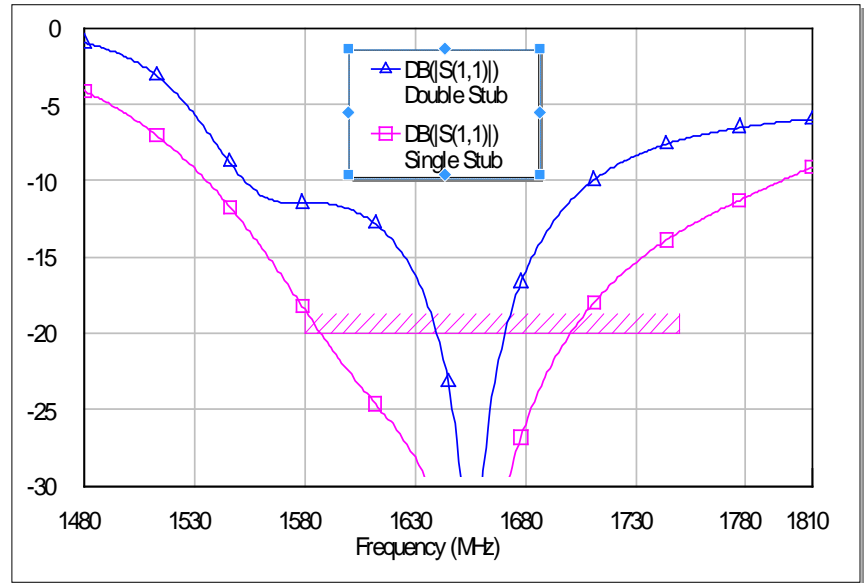


With a shunt stub (B=0.01018 S)

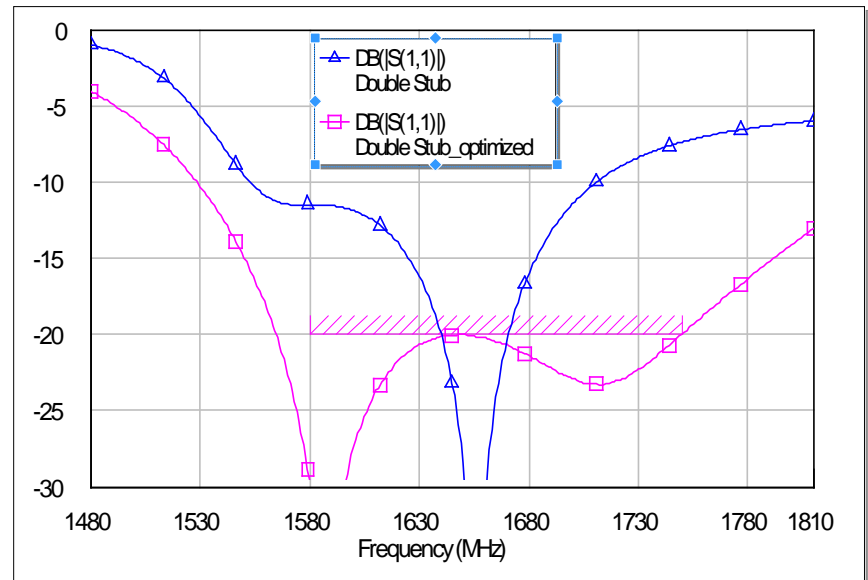
# 3 Elements network



Initially synthesized as double-stub:  
 $\theta_1=34.05^\circ$ ,  $\theta_2=26.57$ ,  $\theta_3=135^\circ$

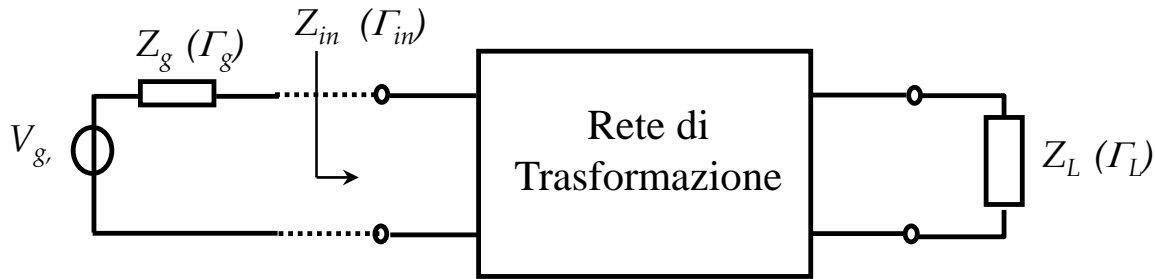


Optimized network:  
 $\theta_1=89.91^\circ$ ,  $\theta_2=72.47$ ,  $\theta_3=105.3^\circ$





# Impedance Transformation

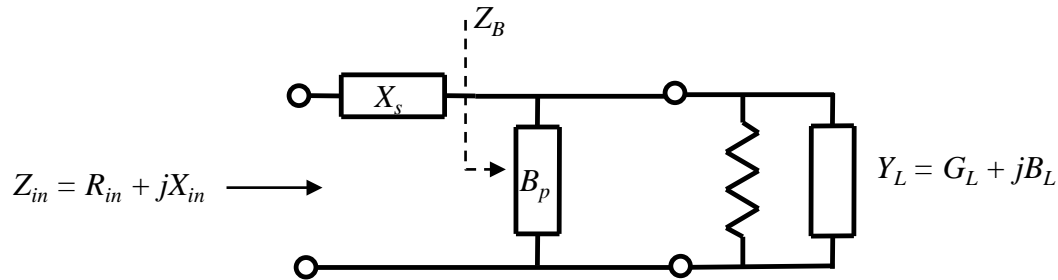


Goal: given  $Z_L$  e  $Z_{goal}$  find the network parameters for which  $Z_{in} = Z_{goal}$

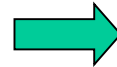
$$\text{Conjugate matching} \rightarrow Z_{goal} = Z_g^*, \quad \Gamma_{goal} = \Gamma_g^*$$

Usually the elements of the network are lossless (no power lost in the transformation)

## Lumped-element network (1)



$$Z_{in} = R_{in} + jX_{in} = jX_s + \frac{1}{G_L + j(B_p + B_L)}$$



$$R_{in} = \frac{G_L}{G_L^2 + (B_p + B_L)^2}$$

$$X_{in} = X_s - \frac{(B_p + B_L)}{G_L^2 + (B_p + B_L)^2}$$

$X_s$  e  $B_p$  are obtained by solving the previous system:

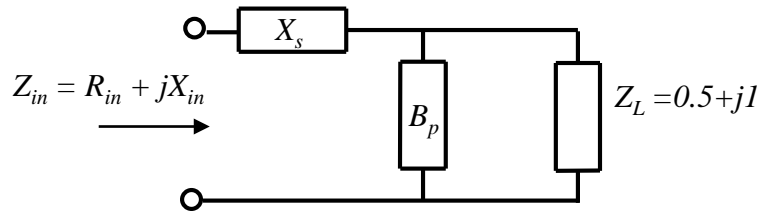
$$B_p = \pm \sqrt{G_L (1/R_{in} - G_L)} - B_L$$

$$X_s = \pm \frac{R_{in}}{G_L} \left[ \sqrt{G_L (1/R_{in} - G_L)} \right] + X_{in}$$

Necessary  
Condition :

$$R_{in} G_L < 1$$

# Example



$$Y_L = \frac{1}{Z_L} = 0.4 - j0.8, \quad Z_{in} = 1 + j1$$

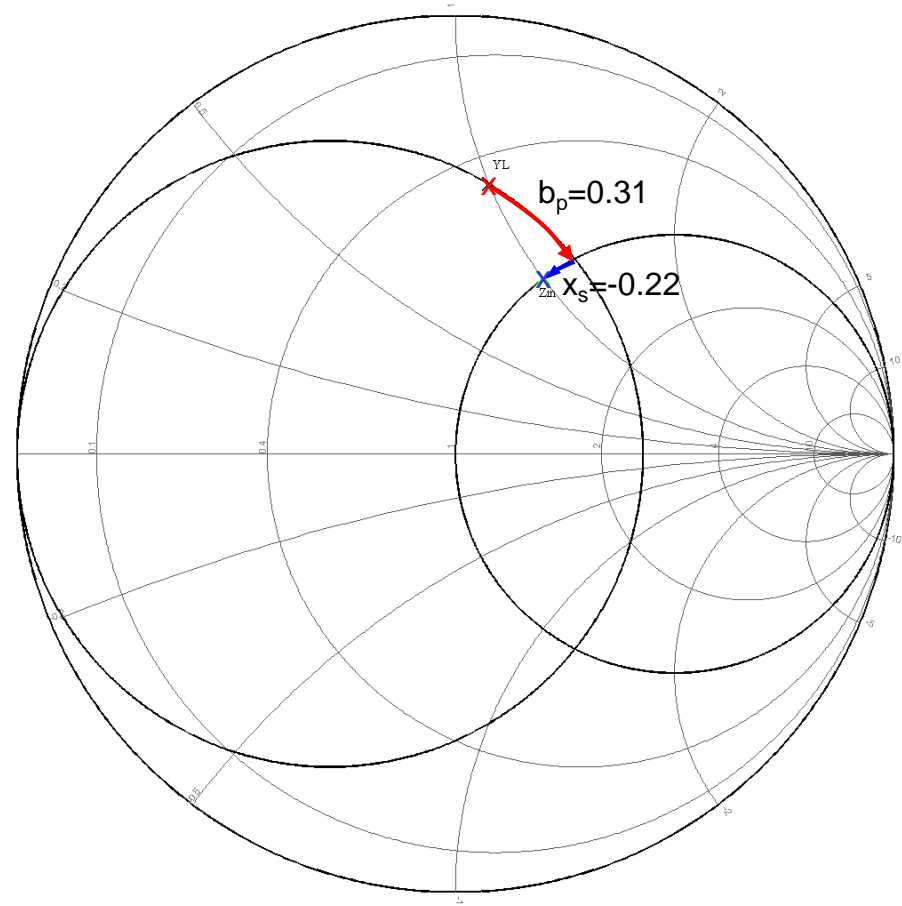
$$G_L \cdot R_{in} < 1$$

$$B_p = \pm \sqrt{G_L (1 - G_L)} - B_L = 1.2899 (0.31)$$

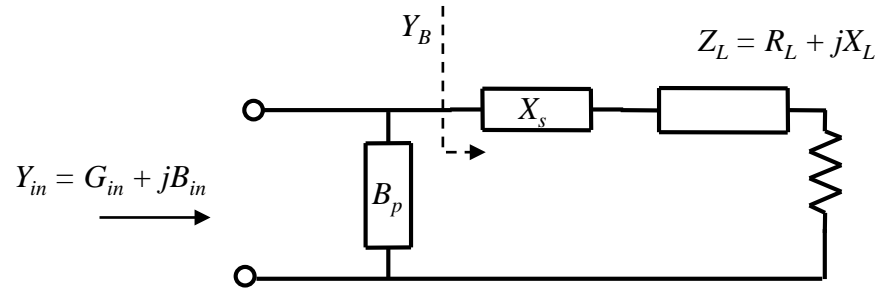
$$X_s = \pm \frac{1}{G_L} \left[ \sqrt{G_L (1 - G_L)} \right] + X_{in} = 2.2247 (-0.2247)$$

## Smith chart solution

1. Draw the circle  $g=G_L/G_0$
2. Draw the circle  $r=R_{in}/R_0$
3. Input  $Y_L/G_0$  and store
4. Select as current point the intersection of the two circles
5.  $b_p$  (normalized) is found in the DeltaY tab
6. Store the current point
7. Input  $Z_{in}$
8.  $x_s$  (normalized) is found in the DeltaZ tab
9. Denormalize  $x_s$  and  $b_p$



## Lumped-element network (2)



$$Y_{in} = G_{in} + jB_{in} = jB_p + \frac{1}{R_L + j(X_s + X_L)}$$



$$G_{in} = \frac{R_L}{R_L^2 + (X_s + X_L)^2}$$

$$B_{in} = B_p - \frac{(X_s + X_L)}{R_L^2 + (X_s + X_L)^2}$$

$X_s$  e  $B_p$  are obtained by solving the previous system :

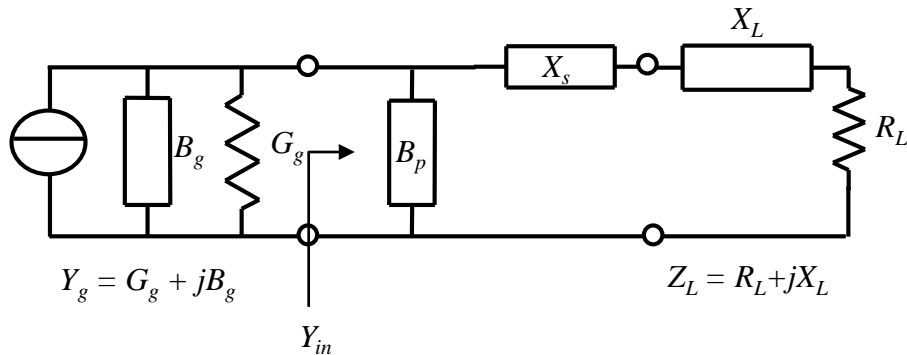
$$X_s = \pm \sqrt{R_L (1/G_{in} - R_L)} - X_L$$

$$B_p = \pm \frac{G_{in}}{R_L} \left[ \sqrt{R_L (1/G_{in} - R_L)} \right] + B_{in}$$

Necessary  
Condition :

$$R_L G_{in} < 1$$

## Example: conjugate matching



$$Z_L = 0.5 + j1$$

$$Y_G = 1 + j2$$

$$Y_{in} = Y_g^* = 1 - j2$$

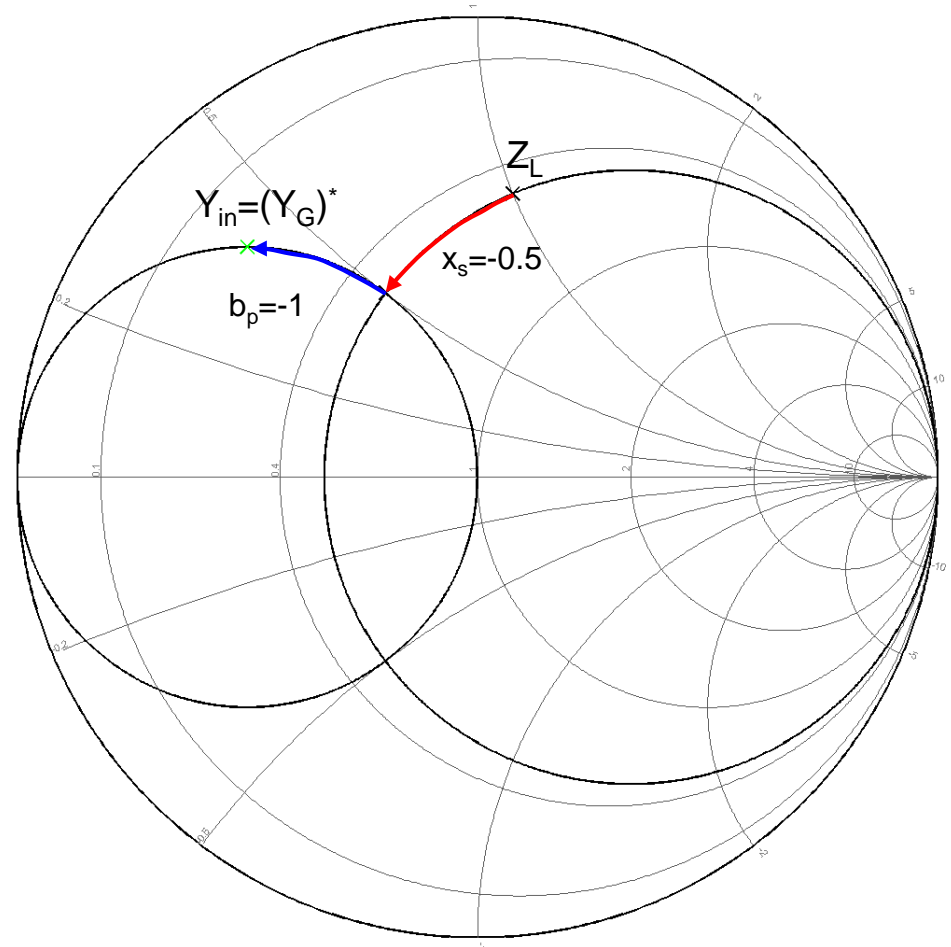
$$R_L \cdot G_g < 1$$

$$X_s = \pm \sqrt{R_L (1 - R_L)} - X_L = -0.5 \quad (-1.5)$$

$$B_p = \pm \frac{1}{R_L} \left[ \sqrt{R_L (1 - R_L)} \right] + B_{in} = -3 \quad (-1)$$

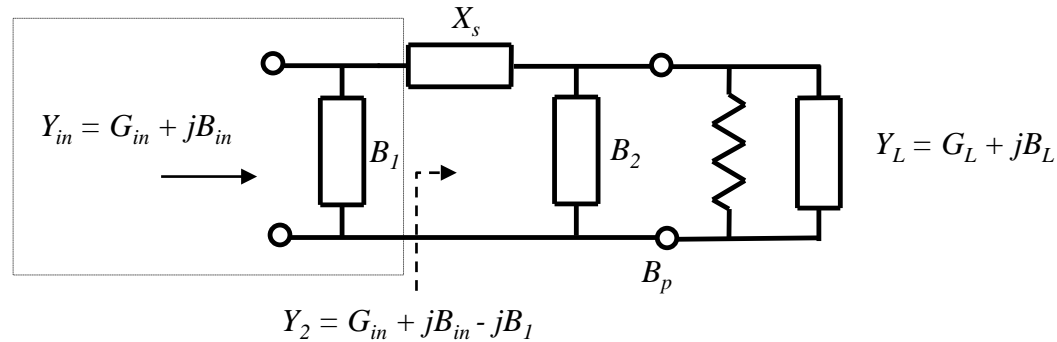
# Smith chart solution

1. Draw the circle  $r=R_L/R_0$
2. Draw the circle  $g=G_{in}/G_0$
3. Input  $Z_L/R_0$  and store
4. Select as current point the intersection of the two circles
5.  $X_s$  (normalized) is found in the DeltaZ tab
6. Store the current point
7. Input  $Y_{in}$
8.  $b_p$  (normalized) is found in the DeltaY tab
9. Denormalize  $x_s$  and  $b_p$



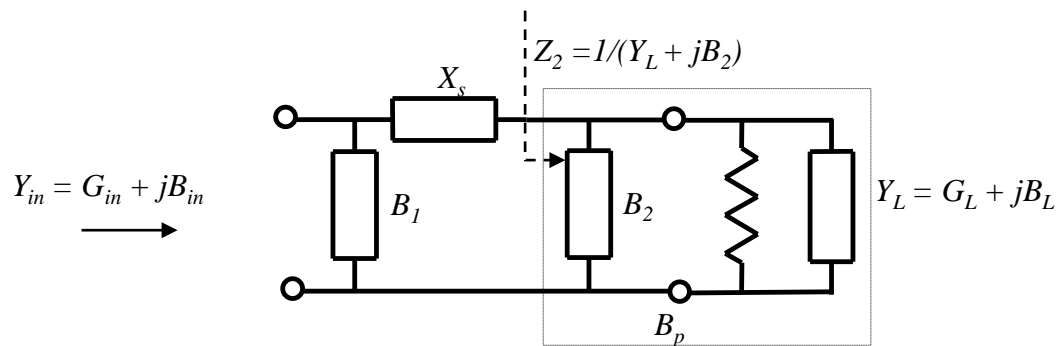
# Π-Network

Being a further element available (with respect to the L-Networks) it is possible to assign one of the shunt components ( $B_1$  o  $B_2$ )



$$\operatorname{Re}\{1/Y_2\} G_L < 1$$

$X_s$  e  $B_2$  are obtained with the formulas of network (1)



$$G_{in} \operatorname{Re}\{Z_2\} < 1$$

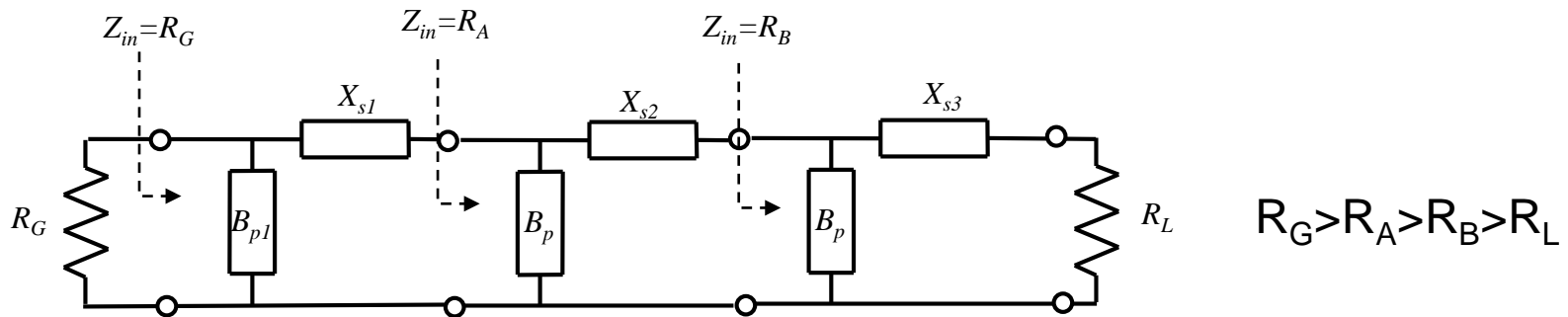
$X_s$  e  $B_1$  are obtained with the formulas of network (2)

The T-network can be designed in a similar way



# Increase the matching bandwidth

If the impedances  $Z_L$  and  $Z_G$  are real, the matching bandwidth is increased by using several L-networks as shown in the following picture:



The formulas seen previously are used for the network design (with  $X_L$ ,  $X_{in}=0$ ). To verify the actual frequency variation of the matching, the network elements must be implemented with real components (capacitors and/or inductors or stubs)

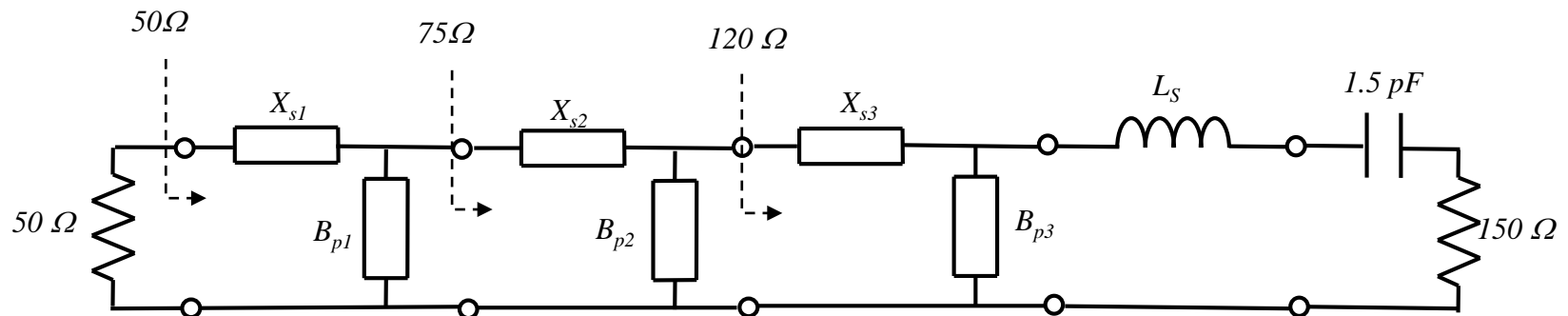
# Example: synthesis + optimization

## Problem

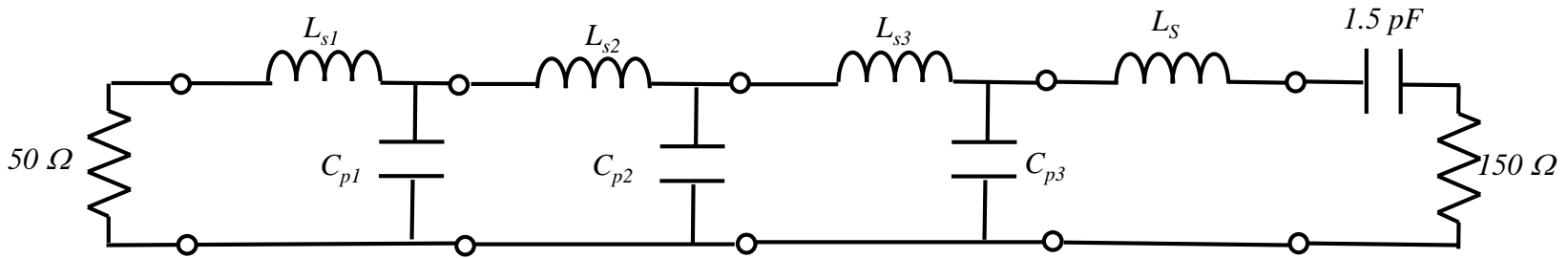
A load is constituted by capacitor (1.5 pF) in series with a resistance of 150 Ohm. Design a lumped elements network for matching this load to 50 Ohm in the band 600-1100 MHz with at least 24 dB of return loss

## First step: the synthesis

A cascade of three L-matching networks is used. The load is first resonated at the center band frequency by means of an inductor, then the 150 Ohm load is transformed into 50 Ohm with  $R_{s1}=120\Omega$  and  $R_{s2}=75\Omega$ :



# Synthesized Network



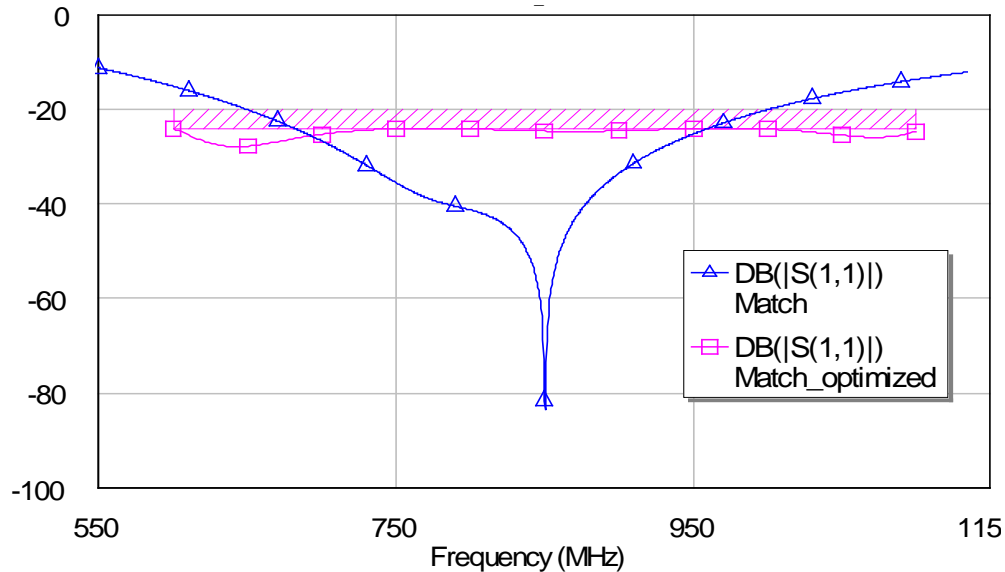
Computed parameters:

Initial (synthesis):

$L_{s1}=6.62$      $L_{s2}=10.88$      $L_{s3}=11.23$   
 $C_{p1}=1.7653$     $C_{p2}=1.209$      $C_{p3}=0.6241$   
 $LS=23.37$

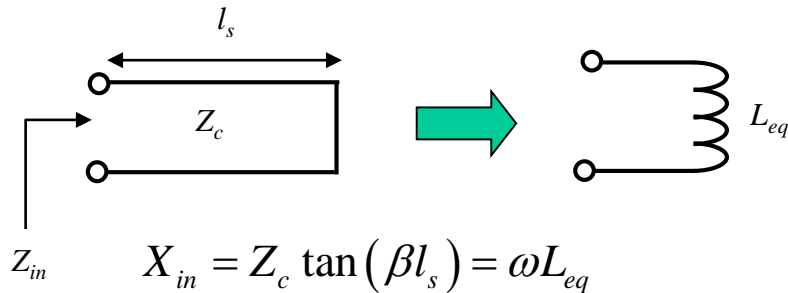
Optimized:

$L_{s1}=7.219$      $L_{s2}=20.58$      $L_{s3}=8.26$   
 $C_{p1}=2.952$     $C_{p2}=1.441$      $C_{p3}=0.4535$   
 $LS=29.61$



# Replace lumped elements with stubs (1)

Approximate equivalence:



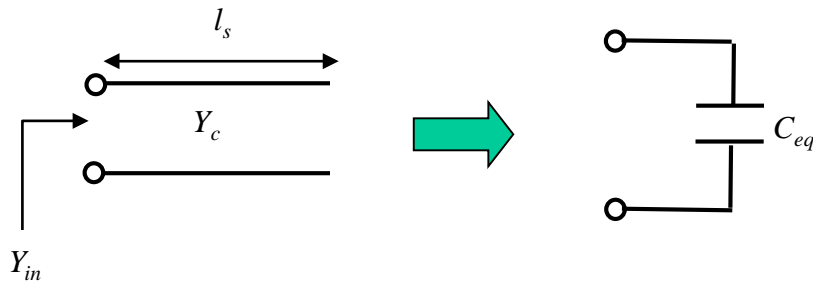
$$X_{in} = Z_c \tan(\beta l_s) = \omega L_{eq}$$

$$Z_c = \frac{\omega_0 L_{eq}}{\tan(\beta_0 l_s)}$$

$$Z_{in} = jZ_c \tan(\beta l_s) \Rightarrow \beta l_s \cong 0$$

$$\Rightarrow Z_{in} \cong jZ_c \beta l_s = j\omega \frac{Z_c l_s}{v}$$

$$L_{eq} \cong \frac{Z_c l_s}{v}$$



$$B_{in} = Y_c \tan(\beta l_s) = \omega C_{eq}$$

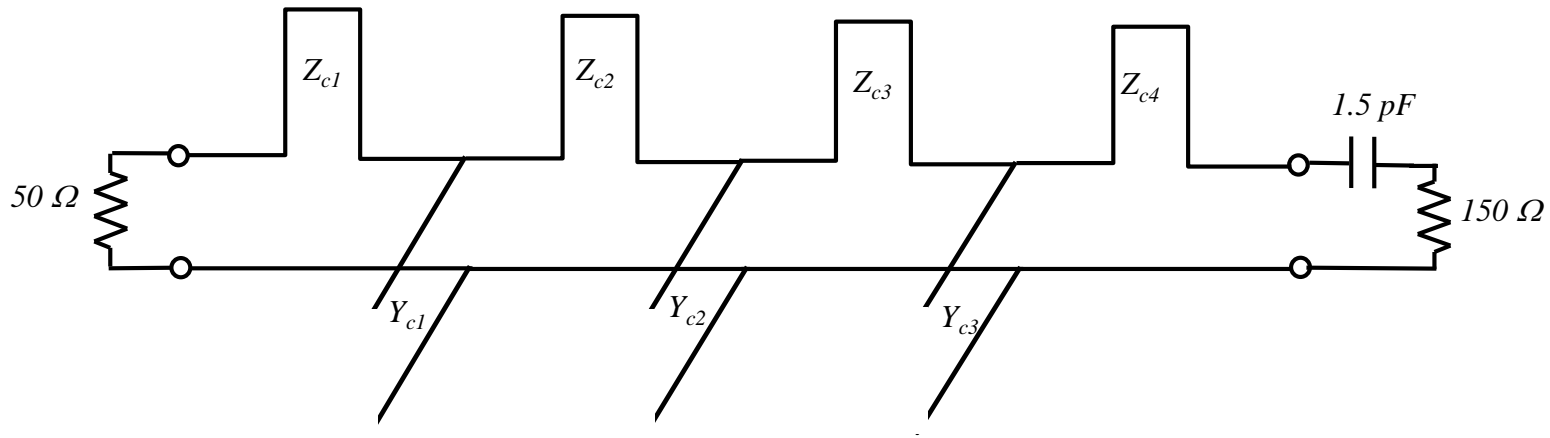
$$Y_c = \frac{\omega_0 C_{eq}}{\tan(\beta_0 l_s)}$$

$$Y_{in} = jY_c \tan(\beta l_s) \Rightarrow \beta l_s \cong 0$$

$$\Rightarrow Y_{in} \cong jY_c \beta l_s = j\omega \frac{Y_c l_s}{v}$$

$$C_{eq} \cong \frac{Y_c l_s}{v}$$

## Equivalent network



Assigned lengths:

$$\beta l_s = 20^\circ \text{ at } f_0 = 850 \text{ MHz with } \epsilon_r = 2$$

$$\rightarrow l_s = 13.86 \text{ mm}$$

Computed  $Z_c$ :

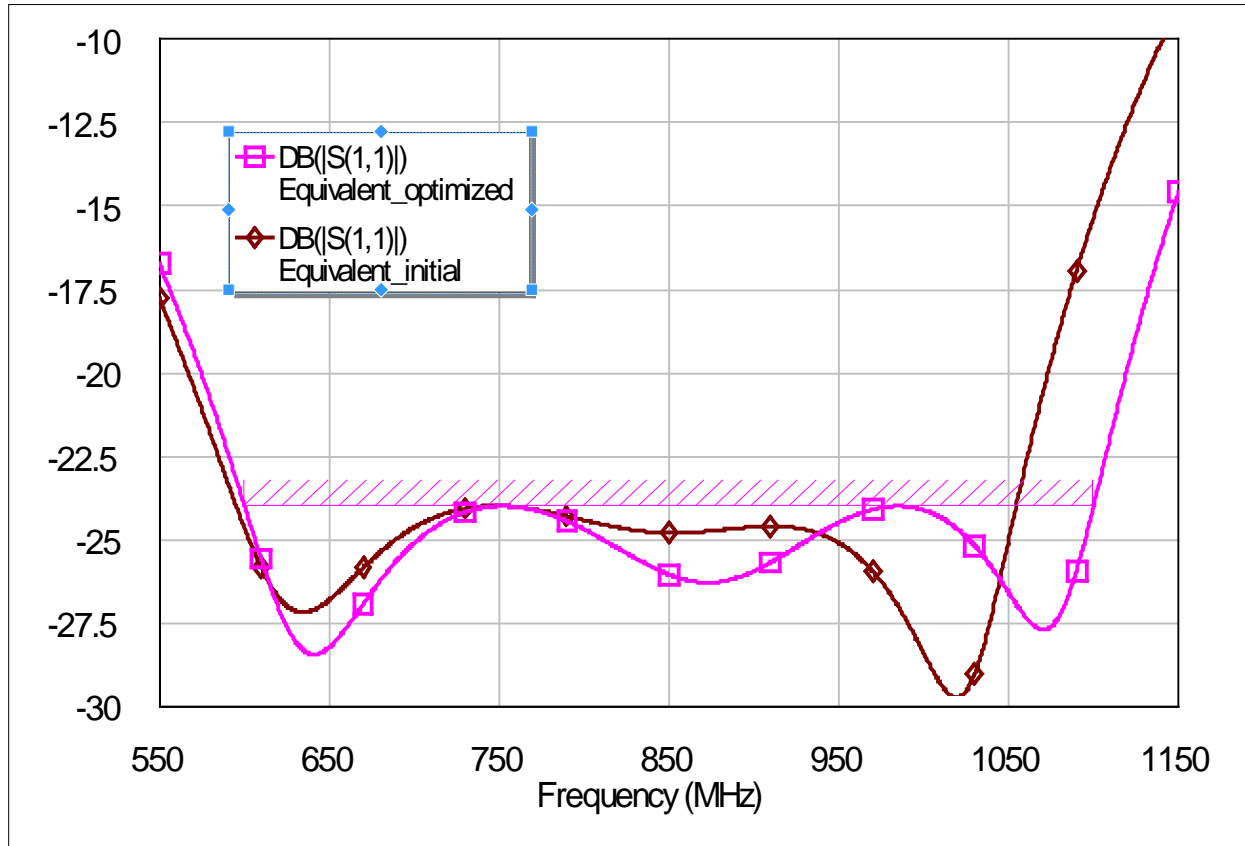
$$Z_{c1} = 110.49, Z_{c2} = 314.98, Z_{c3} = 126.42,$$

$$Z_{c4} = 453.19$$

Computed  $Y_c$ :

$$Y_{c1} = 0.0452, Y_{c2} = 0.0221, Y_{c3} = 0.0067$$

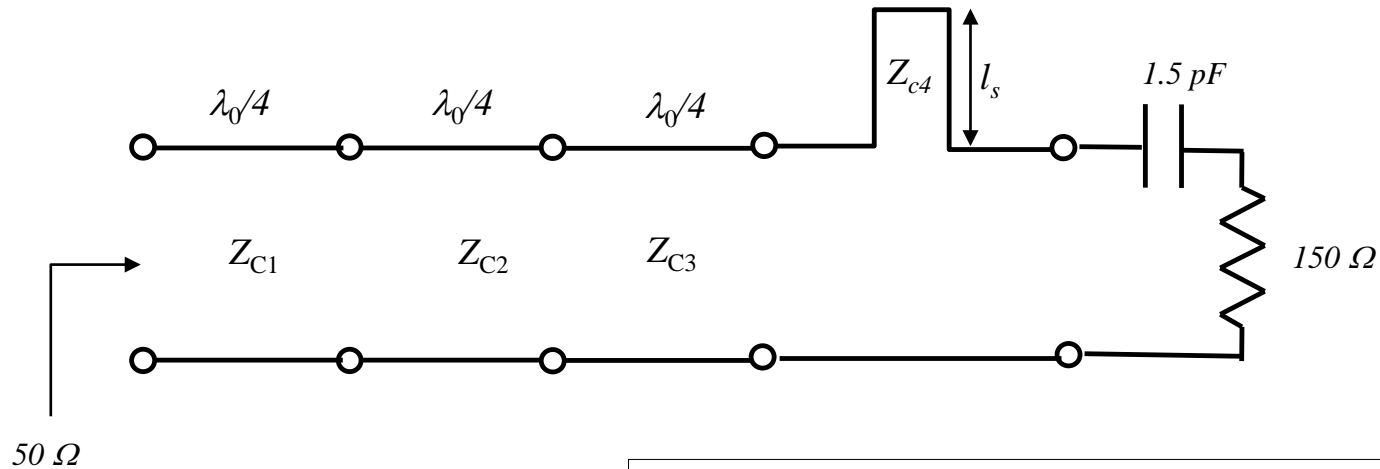
# Optimized response



Zc1=113.1  
Zc2=321.1  
Zc3=63.63  
Zc4=460.1

Zp1=23.59  
Zp2=48.75  
Zp3=208.7

# Another distributed solution



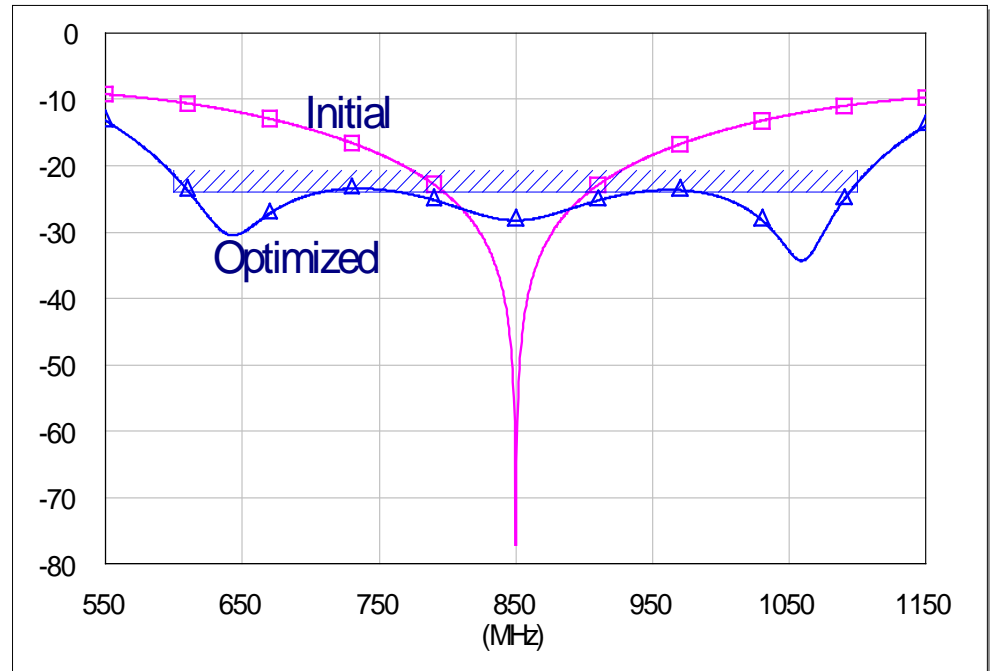
$$Z_{c1} = 50 \cdot 3^{1/8} = 57.36 \Omega$$

$$Z_{c2} = Z_{c1} \cdot 3^{3/8} = 86.6 \Omega$$

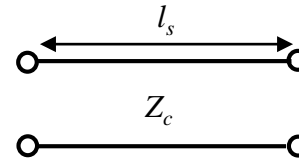
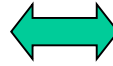
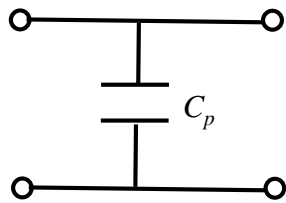
$$Z_{c3} = Z_{c2} \cdot 3^{3/8} = 130.75 \Omega$$

$$l_s = 27.57 \text{ mm} \quad (\beta l_s = 39.77^\circ)$$

$$Z_{c4} = 150 \Omega \quad (L_{\text{eq}} = 23.1 \text{ nH})$$



Approximate equivalence:



$$Z_C = \begin{vmatrix} \frac{1}{j\omega C_p} & -\frac{1}{j\omega C_p} \\ -\frac{1}{j\omega C_p} & \frac{1}{j\omega C_p} \end{vmatrix}$$

$$Z_S = \begin{vmatrix} \frac{1}{jY_c \tan(\beta l_s)} & -\frac{1}{jY_c \sin(\beta l_s)} \\ -\frac{1}{jY_c \sin(\beta l_s)} & \frac{1}{jY_c \tan(\beta l_s)} \end{vmatrix}$$

$$Z_L \cong Z_S \Rightarrow \beta l_s \cong 0 \Rightarrow \tan(\beta l_s) \cong \sin(\beta l_s) \cong \beta l_s$$

$$\omega C_p \cong Y_c \beta l_s$$

$$C_s \cong \frac{Y_c l_s}{v}$$